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CALCULATION OF THE FLAPWISE BENDING, EDGEWISE BENDING, AND TORSIONAL VIBRATIONS OF ROTOR BLADES WITH COUPLED NATURAL MODES AND FREQUENCIES

H. Oette

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16. Abstract The natural-mode-method, presented by Guillenschmidt, is expanded here from one to two degrees of freedom, and then to the three coupled degrees of flapwise bending, edgewise bending and torsion. The partial differential equations of motion are taken from Houbolt and Brooks. For the coupled natural modes which are assumed as known the orthogonal relations are formulated and the differential equations for the generalized degrees of freedom are derived. The theory is further extended by adding the moment of inertia due to the extension of the blade across to its axis and applying a flapping coordinate system. For the numerical examples the following items are taken into account: The forward flight data of the S-61 helicopter, coupled natural modes computed by the multihinge articulated blade method, and schematized nonstationary aerodynamic coefficient curves.

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$'$ = d/dx = Differentiation by x

$\dot{}$ = d/dt = Differentiation with respect to time

x, y, z = Coordinate system of rotor-blade vibrations

y_E, z_E = Deflections of elastic axis (straight in undeformed state) in y and z directions

ϑ_U = Profile angle of attack in undeformed state

ϑ_E = Additional angle of attack due to twisting

a = Flapping hinge distance

R = Rotor radius (for $\beta_{B1} = 0$)

R_A = R - a

β_{B1} = Flapping angle

e_A = Distance from rotor axis up to elastic axis

e_F = Distance from elastic axis up to stress axis

l_{ES} = Distance from elastic axis back to profile center of gravity

i_F^2 = = polar radius of gyration (area integral) of a profile; only tension-bearing parts of cross section taken into account

m' = Mass per unit of length in longitudinal direction of blade

$$\left.\begin{aligned} i_{m\zeta}^2 &= \int_F \eta^2\, dm / m \\ i_{m\eta}^2 &= \int_F \zeta^2\, dm / m \end{aligned}\right\}$$ Axial radii of gyration (mass integral) of a profile or blade cross section

$i_m^2 = i_{m\eta}^2 + i_{m\zeta}^2$ = Polar radius of gyration (mass integral)

E = Modulus of elasticity

G = Shear modulus

I_1 = $\int \zeta^2 dF$ = First principal moment of inertia of a cross section; cf. i_F

I_2 = $\int \eta(\eta - e_F)\, dF$ = second principal moment of inertia

B_1 = $\int(\eta^2 + \zeta^2)(\eta^2 + \zeta^2 - i_F^2)\, dF$

B_2 = $\int(\eta^2 + \zeta^2)(\eta - e_F)\, dF$

J = St. Venant's torsion moment of inertia

Y' = Aerodynamic force in y direction per unit of length of blade

Z' = Aerodynamic force in z direction per unit of length of blade

M' = Aerodynamic moment about the elastic axis per unit of length

M_y, M_z = Bending moment about y and z axes

Q_y, Q_z = Shear force in y and z directions

I_c I_o I_s = Abbreviations, see Eq. (3.2)

f g h = Abbreviations, see remark following Eq. (4.23)

Y_e' Z_e' M_e' = Abbreviations, see Eq. (5.31)

Y_{ee}' Z_{ee}' M_{ee}' = Abbreviations, see Eq. (6.5)

i_{mc} i_{mo} i_{ms} = Abbreviations, see Eq. (6.7)

P_{xF} = Force in radial direction due to all centrifugal forces between x and blade tip

ω_{Ro} = Angular velocity of rotor rotation

ν_j = Angular velocity of j-th natural vibration

q_j = Time function of j-th natural mode in forced vibration

y_{Ej}^* z_{Ej}^* ϑ_{Ej}^* = j-th natural vibration

$\bar{y}_{Ej}$ $\bar{z}_{Ej}$ $\bar{\vartheta}_{Ej}$ = j-th natural mode

i, j, p, q, n = Counting indices

z = Number of rotor blades

l = Blade chord

$\vartheta_{0,7}$ = θ_u at $x = 0.7(R - a)$, without cyclic blade control

$\Delta\vartheta$ = Measure of linear blade twist (difference between θ_u at blade tip and θ_u at blade root in purely linear twist)

ϑ_E, ϑ_S = cos ψ_{B1}- and sin ψ_{B1}- components of cyclic blade control

ψ_{Bl} = Blade azimuthal angle

v_z = Flight velocity in direction of rotor axis, positive downward

w_i = Induced wind speed (downwash) in the rotor plane, in the direction of the rotor axis, positive downward

v_x = Flight velocity perpendicular to rotor axis, positive forward

B = The abscissa $x|(R - a)$, out to which lift is to be present (measure of peripheral drop)

ρ = Air density

NU = Number of rotations about rotor axis

1.M_β (2.M_β) = First (second) natural flapping mode

1.M_ζ (2.M_ζ) = First (second) natural swiveling mode

1.M_T (2.M_T) = First (second) natural torsion mode

NB = 0(1): the coordinate system is (is not) rotated through the flapping angle $\beta_{B1}(t)$ calculated in advance

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H. Oette,
German Research and Experimental Laboratory for Air and Space Flight, Institute for Rotary Wing Aircraft, Stuttgart

1. Introduction /7*

The natural mode method is particularly well suited for calculating force vibrations of complicated systems. In this method, the deflection at any time is the result of superimposing different natural modes, each with its own individual factor q_i, $i = 1, \ldots, n$. Through appropriate selection of q_1 through q_n, any arbitrary deflection can be represented (for $n \to \infty$) or approximated, the precision increasing rapidly with n.

Hence, the degrees of freedom possessed by the vibrating system are to be the time-dependent functions c (t). In other words, the degrees of freedom are transformed by a rule $(y_E, z_E, \theta_E)(x,t) \to q(i,t)$. The natural modes depending only on location, which consists of the three functions $y_{Ei}(x)$, $z_{Ei}(x)$, and $\theta_{Ei}(x)$ for rotor vibrations in the flapwise, edgewise and torsion directions, and the associated natural frequencies ν_i are assumed to be known. They are best calculated by a segment method (Myklestad method, multihinge articulated blade technique, matrix theory of statics and dynamics). To each natural mode is assigned a generalized mass (see the denominators of Equations (5.8), (5.21), (5.29), etc.), which can also be taken as known, since it can be calculated in advance with the aid of the natural modes. The natural modes, natural frequencies, and generalized masses incorporate all the mechanical properties of the blade, such as rigidities, mass distribution, rate of revolution of the rotor, boundary conditions, built-in torsion, etc. in compressed (transformed) form, so that the vibration calculation is correspondingly simple.

If the forced oscillations were to be calculated with one of the segment methods, the number of degrees of freedom which would have to be taken into account in order to obtain the same accuracy would be much larger than the number n mentioned above. This number would be at least equal to the number of segments, multiplied by the number of degrees of freedom, i.e. by 3 in the case of flapwise bending, edgewise bending, and torsion.

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* Numbers in the margin indicate pagination in the foreign text.

The calculation of forced oscillations using the Galerkin method or related techniques is of less interest these days, since the natural modes and frequencies can be calculated, and since the natural-mode method works quite well in the present work, even when the problems are very difficult. Hence, in a vibration calculation, the Galerkin method should require more numerical computations and provide less accurate results, even though the number of input functions is the same. /8

2. Symbols [see pp. iv-vi]

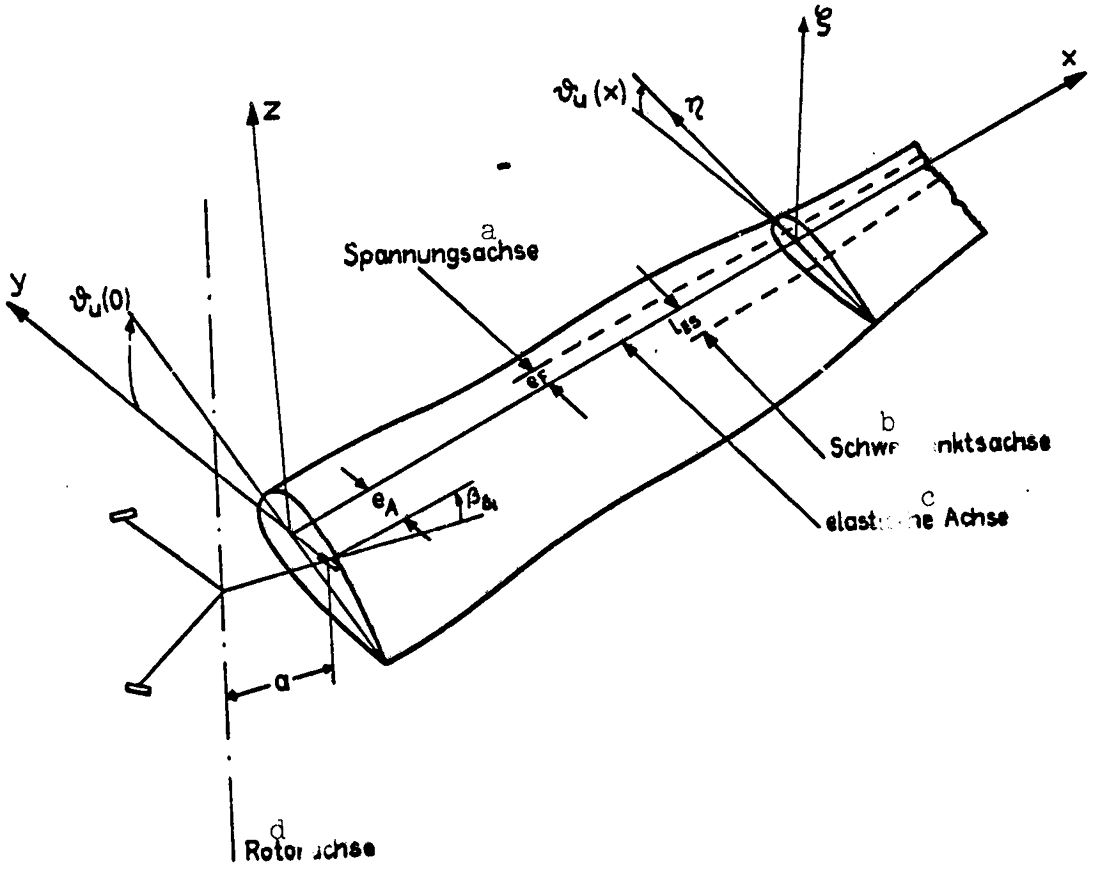

Fig. 2.1. Geometry of the rotor blade.

Key: a. Stress axis
b. Center-of-gravity axis
c. Elastic axis
d. Rotor axis

3. Differential Equations and Natural Vibrations /12

3.1. Fundamental Differential Equations

We wish to treat different types of differential equations. We will demonstrate the basic features of the natural-mode method using the differential equation for pure flapwise bending

$$(EI\,z_E'')'' - (P_{xF}\,z_E')' + m'\ddot{z}_E = Z'(x,t,\dot{z}_E) \tag{3.1}$$

The transition to coupled vibrations can be conveniently depicted using coupled flapwise and edgewise bending

$$\begin{aligned}
&(EI_c\,y_E'')'' + (EI_0\,z_E'')'' - (P_{xF}\,y_E')' + m'\ddot{y}_E - \omega_{R0}^2 m' y_E = Y'(x,t,\dot{z}_E)\\
&(EI_0\,y_E'')'' + (EI_s\,z_E'')'' - (P_{xF}\,z_E')' + m'\ddot{z}_E = Z'(x,t,\dot{z}_E)\\
&EI_c = EI_1 \sin^2\vartheta_u + EI_2\cos^2\vartheta_u \qquad EI_0 = (EI_2 - EI_1)\sin\vartheta_u\cos\vartheta_u\\
&EI_s = EI_1\cos^2\vartheta_u + EI_2\sin^2\vartheta_u
\end{aligned} \tag{3.2}$$

The flatter calculation uses the flapwise-bending and torsion degrees of freedom in general. Therefore, we also work with

$$\begin{aligned}
&(EI z_E'')'' - (P_{xF} e_F \vartheta_E \cos\vartheta_u)'' - (P_{xF} z_E')' + m'\ddot{z}_E - m' l_{ES}\ddot{\vartheta}_E\cos\vartheta_u = Z'(x,t,\dot{z}_E,\vartheta_E,\dot{\vartheta}_E)\\
&-[(GJ + P_{xF} i_F^2)\,\vartheta_E']' - P_{xF} e_F z_E''\cos\vartheta_u + \omega_{R0}^2 m'(i_{m\xi}^2 - i_{m\eta}^2)\vartheta_E\cos 2\vartheta_u\\
&+ m' i_m^2\ddot{\vartheta}_E - m' l_{ES}\ddot{z}_E\cos\vartheta_u = M'(x,t,\dot{z}_E,\vartheta_E,\dot{\vartheta}_E)
\end{aligned} \tag{3.3}$$

Finally, the most precise representation -- which, however, is also the most arduous one -- with the three degrees of freedom y_E, z_E and θ_E will be considered:

$$
\begin{aligned}
&\left[EI_c y_E'' + EI_0 z_E'' + P_{xF} e_F \vartheta_E \sin\vartheta_u - EB_2 \vartheta_u' \vartheta_E' \cos\vartheta_u\right]'' - (P_{xF} y_E')' \\
&- \omega_{R0}^2 \left[m' l_{ES}(x+a)\vartheta_E \sin\vartheta_u\right]' - \omega_{R0}^2 m' l_{ES} \vartheta_E \sin\vartheta_u - \omega_{R0}^2 m' y_E + m' \ddot{y}_E + m' l_{ES} \ddot{\vartheta}_E \sin\vartheta_u \\
&= Y'(x, t, \dot{y}_E, \dot{z}_E, \vartheta_E) + (P_{xF} e_F \cos\vartheta_u)'' - \omega_{R0}^2 \left[m' l_{ES}(x+a)\cos\vartheta_u\right]' + \omega_{R0}^2 m'(e_A - l_{ES}\cos\vartheta_u)
\end{aligned}
$$

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$$
\begin{aligned}
&\left[EI_0 y_E'' + EI_S z_E'' - P_{xF} e_F \vartheta_E \cos\vartheta_u - EB_2 \vartheta_u' \vartheta_E' \sin\vartheta_u\right]'' - (P_{xF} z_E')' \\
&+ \omega_{R0}^2 \left[m' l_{ES}(x+a)\vartheta_E \cos\vartheta_u\right]' + m' \ddot{z}_E - m' l_{ES} \ddot{\vartheta}_E \cos\vartheta_u \\
&= Z'(x, t, \dot{y}_E, \dot{z}_E, \vartheta_E, \dot{\vartheta}_E) + (P_{xF} e_F \sin\vartheta_u)'' - \omega_{R0}^2 \left[m' l_{ES}(x+a)\sin\vartheta_u\right]'
\end{aligned}
\tag{3.4}
$$

$$
\begin{aligned}
&-\left[(GJ + P_{xF} i_F^2 + EB_1 \vartheta_u'^2)\vartheta_E' - EB_2 \vartheta_u' (y_E'' \cos\vartheta_u + z_E'' \sin\vartheta_u)\right]' \\
&- P_{xF} e_F (z_E'' \cos\vartheta_u - y_E'' \sin\vartheta_u) - \omega_{R0}^2 m' l_{ES}(x+a)(z_E' \cos\vartheta_u - y_E' \sin\vartheta_u) - \omega_{R0}^2 m' l_{ES} y_E \sin\vartheta_u \\
&+ \omega_{R0}^2 m' \left[(i_{m\xi}^2 - i_{m\eta}^2)\cos 2\vartheta_u - l_{ES} e_A \cos\vartheta_u\right]\vartheta_E + m' i_m^2 \ddot{\vartheta}_E + m' l_{ES}(\ddot{y}_E \sin\vartheta_u - \ddot{z}_E \cos\vartheta_u) \\
&= M'(x, t, \dot{y}_E, \dot{z}_E, \vartheta_E, \dot{\vartheta}_E) + (P_{xF} i_F^2 \vartheta_u')' - \omega_{R0}^2 m' \left[(i_{m\xi}^2 - i_{m\eta}^2)\sin\vartheta_u \cos\vartheta_u - l_{ES} e_A \sin\vartheta_u\right]
\end{aligned}
$$

Equations (3.1) through (3.4) are taken from the work of Houbolt-Brooks [1], where x has been replaced by (x + a) because we start the x axis at the flapping hinge. In Chapter 6, we will deal with a system of equations which is even more general than (3.4). In Equations (3.1) through (3.3), the unused degrees of freedom are set equal to zero, along with some smaller quantities, when neglecting the latter is consistent with neglecting the particular degrees of freedom. Let Y', Z' and M' be the excitation functions. Y',Z' = force in y and z directions per unit of length, and M' = moment about the x axis per unit of length. They are to contain the aerodynamic loads including the changes in the latter induced by blade vibrations. If needed, the inertial forces due to the gravitational acceleration g and the motion of the x-y-z coordinate system can also be included, e.g. when the latter accompanies the flapping motion (see Ch. 6).

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The original unknowns y_E, z_E and θ_E still appear in the excitation functions. This is because there is no point in putting them all on the left side and incorporating them in the natural-vibration calculation, even in the rare cases in which this is possible. For this reason, Y', Z' and M' cannot really be called perturbation functions. Nevertheless, the unknowns on the right side are still dependent variables, functions of the q_i. For example, the Runge-Kutta method for stepwise solution of differential equations works not only (cf. Eq. (5.8)) for $\ddot{q}_i = -\nu_i^2 a_i + f(t)$, but also for $\ddot{q}_i = -\nu_i^2 q_i + f(t, q_1, q_2, \ldots, \dot{q}_1, \dot{q}_2, \ldots)$. Iteration also works for $f(\ldots \ddot{q}_1, \ddot{q}_2, \ldots)$. Iterative solutions are possible even for $f(\ldots \dddot{q}_1, \dddot{q}_2, \ldots)$ and higher derivatives, when the influence of $\dddot{q}$ etc. is sufficiently small.

Strictly speaking, the unknowns influence the excitation functions Y', Z' and M' in very many ways. However, in our description of the functional relationships, we will stick to the most important influences. For example, in Eq. (3.2), $\dot{z}_E$ occurs in Z' and Y' because of its influence on the angle α of attack. Although they play a certain role via their influence on the relative airspeed or on $\dot{\alpha}$, the variables $\dot{y}_E$ and $\ddot{z}_E$ are suppressed.

Aside from the excitation functions, certain other terms occur on the right side of Eq. (3.4). They come from the centrifugal force field, and induce a kind of prestress on the system with constant y_E, z_E and θ_E values, upon which further deflections are superimposed.

3.2. Natural Vibrations

The natural vibrations corresponding to Eqs. (3.1) through (3.4) can be calculated in approximation, but essentially with arbitrary accuracy, by means of a segment method. The multihinge articulated blade method calculation has been worked out, programmed and tested on numerical examples for all these cases. The computing time on the IBM 1130 of the DFVLR in Stuttgart is only a few minutes in any case. The program delivers the natural modes numerically as sequences of points, through which a solid /15
curve can then be drawn, but this procedure is normally not necessary. Instead, when e.g. the natural mode is to be used in an integration, the integral will be converted to a sum and added up over the sequence of points. Nevertheless, we will continue to speak of functions (written in closed form), integrals, etc. because this simplifies the terminology.

Let the natural vibrations of Eq. (3.1) and thus its solutions for the case in which the right side vanishes, be

$$z_{Ej}^*(x,t) = \bar{z}_{Ej}(x) \cdot e^{i\nu_j t} \qquad j = 1, 2, 3 \ldots \tag{3.5}$$

* = Eigensolution;
‾ = Natural modes;
ν = Natural frequency (real)

Let the solutions of Eq. (3.2) be

$$\begin{bmatrix} y_E \\ \\ z_E \end{bmatrix}_j^* (x,t) = \begin{bmatrix} \bar{y}_E \\ \\ \bar{z}_E \end{bmatrix}_j (x)\cdot e^{i\nu_j t} \qquad j = 1,2,3\ldots \tag{3.6}$$

when the right side vanishes. Since the blade can vibrate in two directions, the eigensolution in this case consists of two functions of x and t, and the natural mode accordingly consists of two functions of x, which we will write as a vector because of their relationship.

Analogously the solutions of Eq. (3.3) when the right side vanishes are

$$\begin{bmatrix} z_E \\ \\ \vartheta_E \end{bmatrix}_j^* (x,t) = \begin{bmatrix} \bar{z}_E \\ \\ \bar{\vartheta}_E \end{bmatrix}_j (x)\cdot e^{i\nu_j t} \qquad j = 1,2,3\ldots \tag{3.7}$$

The solutions of Eq. (3.4) when the right sides vanish are

$$\begin{bmatrix} y_E \\ z_E \\ \vartheta_E \end{bmatrix}_j^* (x,t) = \begin{bmatrix} \bar{y}_E \\ \bar{z}_E \\ \bar{\vartheta}_E \end{bmatrix}_j (x)\cdot e^{i\nu_j t} \qquad j = 1,2,3\ldots \tag{3.8}$$

4. Orthogonality Relations /16

4.1. Orthogonality Relation for Uncoupled Flapwise Bending

Later on in the calculations, we will require some orthogonality relations. We will therefore derive them in order for Eqs. (3.5) through (3.8). z_{ej}^* from Eq. (3.5) satisfies Eq. (3.1) when $Z' = 0$ there, i.e.

$$(EI\, z_{Ej}^{*''})'' - (P_{xF}\, z_{Ej}^{*'})' + m'\ddot{z}_{Ej}^* = 0 \qquad j = 1,2,3\ldots \tag{4.1}$$

Eq. (3.5) then implies

$$(EI\bar{z}_{Ej}'')'' - (P_{xF}\bar{z}_{Ej}')' - \nu_j^2 m'\bar{z}_{Ej} = 0 \qquad j = 1,2,3\ldots \tag{4.2}$$

For j = p, and j = q, Eq. (4.2) becomes

$$\begin{aligned} &(EI\bar{z}_{Ep}'')'' - (P_{xF}\bar{z}_{Ep}')' - \nu_p^2 m'\bar{z}_{Ep} = 0 \\ &(EI\bar{z}_{Eq}'')'' - (P_{xF}\bar{z}_{Eq}')' - \nu_q^2 m'\bar{z}_{Eq} = 0 \end{aligned} \tag{4.3}$$

These equations are multiplied by $\bar{z}_{Eq}$ and $\bar{z}_{Ep}$, and integrated over the length of the blade, and then one is subtracted from the other.

$$\begin{aligned} &\int_0^{R_A}(EI\bar{z}_{Ep}'')''\bar{z}_{Eq}\,dx - \int_0^{R_A}(EI\bar{z}_{Eq}'')''\bar{z}_{Ep}\,dx - \int_0^{R_A}(P_{xF}\bar{z}_{Ep}')'\bar{z}_{Eq}\,dx \\ &+\int_0^{R_A}(P_{xF}\bar{z}_{Eq}')'\bar{z}_{Ep}\,dx = (\nu_p^2-\nu_q^2)\int_0^{R_A} m'\bar{z}_{Ep}\bar{z}_{Eq}\,dx \end{aligned} \tag{4.4}$$

We now wish to show that the left side of Eq. (4.4) is equal to zero. Integrating by parts once and then twice yields

$$\begin{aligned} &\int_0^{R_A} a'b\,dx = \left[ab\right]_0^{R_A} - \int_0^{R_A} ab'dx \\ &\int_0^{R_A} a''b\,dx = \left[a'b\right]_0^{R_A} dx - \int_0^{R_A} a'b'dx = \left[a'b\right]_0^{R_A} - \left[ab'\right]_0^{R_A} + \int_0^{R_A} ab''dx \end{aligned} \tag{4.5}$$

Integrating the first two expressions in Eq. (4.4) by parts twice, and the two following terms once, we obtain /17

$$\begin{aligned} &\left[(EI\bar{z}_{Ep}'')'\bar{z}_{Eq} - EI\bar{z}_{Ep}''\bar{z}_{Eq}' - (EI\bar{z}_{Eq}'')'\bar{z}_{Ep} + EI\bar{z}_{Eq}''\bar{z}_{Ep}'\right]_0^{R_A} \\ &- \left[P_{xF}\bar{z}_{Ep}'\bar{z}_{Eq} - P_{xF}\bar{z}_{Eq}'\bar{z}_{Ep}\right]_0^{R_A} = (\nu_p^2-\nu_q^2)\int_0^{R_A} m'\bar{z}_{Ep}\bar{z}_{Eq}\,dx \end{aligned} \tag{4.6}$$

Now on the left side of the equation remain only expressions the value of which at the root and tip of the blade are to be subtracted from one another. It can be shown that all these values are zero, using the boundary conditions. It is well known that the bending moment and the shear force in the z-direction satisfy

$$M = EIz_E'' \qquad Q = -M' + P_{xF} z_E' = -(EIz_E'')' + P_{xF} z_E' \tag{4.7}$$

M, Q and P_{xF} vanish at the tip of the blade. Therefore, the expressions in brackets in Eq. (4.6) are equal to zero for $x = R_A$. At the root of the blade, z_E and either M or z_E' vanish. Cases apparently incompatible with this are easy to incorporate into the system. For example, if the flapping hinge is spring loaded, the beginning of the blades is simply assumed to be ahead of this hinge (to the left of it in Fig. 2.1). If the hinge is equipped with damping, we omit the damping in this case, but put a corresponding $\dot{z}_E'$-dependent pair of forces on the right side of Eq. (3.1). Hence the expressions on the left in Eq. (4.6) are zero for x = 0 as well, and we obtain

$$\boxed{\int_0^{R_A} m' \bar{z}_{Ep} \bar{z}_{Eq} dx = 0 \qquad \text{for} \qquad \nu_p \neq \nu_q} \tag{4.8}$$

The latter inequality can be replaced by $p \neq q$ unless $\nu_p = \nu_q$ (which is hardly conceivable).

4.2. Orthogonality Relation for Coupled Flapwise and Edgewise Bending /18

In analogous fashion, we wish to formulate an orthogonality relation for the natural modes occurring in Eq. (3.6). By definition, y_{Ej}^* and z_{Ej}^* from Eq. (3.6) satisfy the system of equations

$$\begin{aligned} (EI_C y_{Ej}^{*''})'' + (EI_0 z_{Ej}^{*''})'' - (P_{xF} y_{Ej}^{*'})' + m'\ddot{y}_{Ej}^* - \omega_{R0}^2 m' y_{Ej}^* &= 0 \\ (EI_0 y_{Ej}^{*''})'' + (EI_S z_{Ej}^{*''})'' - (P_{xF} z_{Ej}^{*'})' + m'\ddot{z}_{Ej}^* &= 0 \\ j = 1, 2, 3 \ldots \end{aligned} \tag{4.9}$$

With Eq. (3.6), we obtain

$$
\begin{aligned}
&(EI_c \bar{y}_{Ej}'')'' + (EI_0 \bar{z}_{Ej}'')'' - (P_{xF} \bar{y}_{Ej}')' - (\nu_j^2 + \omega_{R0}^2) m' \bar{y}_{Ej} = 0 \\
&(EI_0 \bar{y}_{Ej}'')'' + (EI_s \bar{z}_{Ej}'')'' - (P_{xF} \bar{z}_{Ej}')' - \nu_j^2 m' \bar{z}_{Ej} = 0 \\
&\qquad j = 1, 2, 3 \ldots
\end{aligned}
\tag{4.10}
$$

For j = p and j = q, (4.10) becomes

$$
\begin{aligned}
&(EI_c \bar{y}_{Ep}'')'' + (EI_0 \bar{z}_{Ep}'')'' - (P_{xF} \bar{y}_{Ep}')' - (\nu_p^2 + \omega_{R0}^2) m' \bar{y}_{Ep} = 0 \\
&(EI_0 \bar{y}_{Ep}'')'' + (EI_s \bar{z}_{Ep}'')'' - (P_{xF} \bar{z}_{Ep}')' - \nu_p^2 m' \bar{z}_{Ep} = 0 \\
&(EI_c \bar{y}_{Eq}'')'' + (EI_0 \bar{z}_{Eq}'')'' - (P_{xF} \bar{y}_{Eq}')' - (\nu_q^2 + \omega_{R0}^2) m' \bar{y}_{Eq} = 0 \\
&(EI_0 \bar{y}_{Eq}'')' + (EI_s \bar{z}_{Eq}'')'' - (P_{xF} \bar{z}_{Eq}')' - \nu_q^2 m' \bar{z}_{Eq} = 0
\end{aligned}
\tag{4.11}
$$

We multiply the first equation with $\bar{y}_{Eq}$, the second with $\bar{z}_{Eq}$, the third with $-\bar{y}_{Ep}$, and the fourth with $-\bar{z}_{Ep}$, then add the four equations, and integrate over the length of the blade.

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$$
\begin{aligned}
&\int_0^{R_A} \Big[(EI_c \bar{y}_{Ep}'')'' \bar{y}_{Eq} + (EI_0 \bar{y}_{Ep}'')'' \bar{z}_{Eq} - (EI_c \bar{y}_{Eq}'')'' \bar{y}_{Ep} - (EI_0 \bar{y}_{Eq}'')'' \bar{z}_{Ep} \\
&+ (EI_0 \bar{z}_{Ep}'')'' \bar{y}_{Eq} + (EI_s \bar{z}_{Ep}'')'' \bar{z}_{Eq} - (EI_0 \bar{z}_{Eq}'')'' \bar{y}_{Ep} - (EI_s \bar{z}_{Eq}'')'' \bar{z}_{Ep} \\
&- (P_{xF} \bar{y}_{Ep}')' \bar{y}_{Eq} - (P_{xF} \bar{z}_{Ep}')' \bar{z}_{Eq} + (P_{xF} \bar{y}_{Eq}') \bar{y}_{Ep} + (P_{xF} \bar{z}_{Eq}')' \bar{z}_{Ep} \Big] dx \\
&= (\nu_p^2 - \nu_q^2) \int_0^{R_A} m' (\bar{y}_{Ep} \bar{y}_{Eq} + \bar{z}_{Ep} \bar{z}_{Eq}) \, dx
\end{aligned}
\tag{4.12}
$$

The terms on the left side will again be integrated once or twice by parts, using Eq. (4.5). The result is

$$
\begin{aligned}
&\Big[(EI_c\bar{y}_{Ep}'')'\bar{y}_{Eq}+(EI_0\bar{y}_{Ep}'')'\bar{z}_{Eq}-(EI_c\bar{y}_{Eq}'')'\bar{y}_{Ep}-(EI_0\bar{y}_{Eq}'')'\bar{z}_{Ep}\\
&+(EI_0\bar{z}_{Ep}'')'\bar{y}_{Eq}+(EI_s\bar{z}_{Ep}'')'\bar{z}_{Eq}-(EI_0\bar{z}_{Eq}'')'\bar{y}_{Ep}-(EI_s\bar{z}_{Eq}'')'\bar{z}_{Ep}\\
&-EI_c\bar{y}_{Ep}''\bar{y}_{Eq}'-EI_0\bar{y}_{Ep}''\bar{z}_{Eq}'+EI_c\bar{y}_{Eq}''\bar{y}_{Ep}'+EI_0\bar{y}_{Eq}''\bar{z}_{Ep}'\\
&-EI_0\bar{z}_{Ep}''\bar{y}_{Eq}'-EI_s\bar{z}_{Ep}''\bar{z}_{Eq}'+EI_0\bar{z}_{Eq}''\bar{y}_{Ep}'+EI_s\bar{z}_{Eq}''\bar{z}_{Ep}' \qquad (4.13)\\
&-P_{xF}\bar{y}_{Ep}'\bar{y}_{Eq}-P_{xF}\bar{z}_{Ep}'\bar{z}_{Eq}+P_{xF}\bar{y}_{Eq}'\bar{y}_{Ep}+P_{xF}\bar{z}_{Eq}'\bar{z}_{Ep}\Big]_0^{R_A}\\
&=(\nu_p^2-\nu_q^2)\int_0^{R_A} m'(\bar{y}_{Ep}\bar{y}_{Eq}+\bar{z}_{Ep}\bar{z}_{Eq})\,dx
\end{aligned}
$$

The integrals on the left side have disappeared. This can be attributed to a certain symmetry in Eq. (4.9). The coupling terms, i.e. the term with $z_{Ej}{}^*$ in the first equation and the term with $y_{Ej}{}^*$ in the second equation have the same form. In problems of this type, this symmetry must always be present, since withdrawn from one type of vibration by a coupling effect must all be delivered to another type of vibration. Now, using the boundary conditions, it can again be shown that the expressions on the left side of Eq. (4.13) disappear. We have taken the formulas for the bending moments and the shear forces from the work of Houbolt and Brooks [1]. Using the abbreviations from Eq. (3.2) and neglecting the terms mentioned in the remark to Eq. (3.4), we obtain: /20

$$
\begin{aligned}
M_z &= EI_c y_E'' + EI_0 z_E''\\
M_y &= EI_0 y_E'' + EI_s z_E''\\
Q_y &= -M_z' + P_{xF} y_E' - \bar{m}_z = -(EI_c y_E'')' - (EI_0 z_E'')' + P_{xF} y_E' - \omega_{R0}^2 m' l_{ES}\, x \cos\vartheta_u \qquad (4.14)\\
Q_z &= -M_y' + P_{xF} z_E' - \bar{m}_y = -(EI_0 y_E'')' - (EI_s z_E'')' + P_{xF} z_E' - \omega_{R0}^2 m' l_{ES}\, x \sin\vartheta_u
\end{aligned}
$$

At the tip of the blade, M_z, M_y, Q_y, Q_z, P_{xF}, and m' vanish. Therefore, the sum of all the expressions in the brackets in Eq. (4.13) is zero for $x = R_A$. At the root of the blade, y_E, z_E, and normally M_z or $y_E{}'$ as well as M_y or $z_E{}^*$ will vanish. Hence, the expressions in the brackets in Eq. (4.13) will sum to zero for $x = 0$ as well, as long as one of the following cases applies: rigid restraint at ends, only flapping hinge, only swivel hinge, and both hinges. Other cases, e.g. one hinge with inclined axis, lead to the same results by a somewhat different route. Eq. (4.13) then becomes

$$\boxed{\int_0^{R_A} m'(\bar{y}_{Ep}\,\bar{y}_{Eq} + \bar{z}_{Ep}\,\bar{z}_{Eq})\,dx = 0 \quad \text{for} \quad \nu_p \neq \nu_q} \tag{4.15}$$

4.3. Orthogonality Relation for Coupled Flapwise Bending and Torsion

The next task is to formulate our orthogonality relation for the natural modes introduced in Eq. (3.7). By definition, $z_{Ej}{}^*$ and $\vartheta_{Ej}{}^*$ satisfy the system of equations

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$$\begin{aligned}
&(EI\,z_{Ej}^{*\,''})'' - (P_{xF}\,e_F\,\vartheta_{Ej}^{*}\cos\vartheta_u)'' - (P_{xF}\,z_{Ej}^{*\,'})' + m'\ddot{z}_{Ej}^{*} - m'l_{ES}\,\ddot{\vartheta}_{Ej}^{*}\cos\vartheta_u = 0 \\
&-[(GJ + P_{xF}\,i_F^2)\,\vartheta_{Ej}^{*\,'}]' - P_{xF}\,e_F\,z_{Ej}^{*\,''}\cos\vartheta_u + \omega_{R0}^2\,m'(i_{m\xi}^2 - i_{m\eta}^2)\,\vartheta_{Ej}^{*}\cos 2\vartheta_u \\
&\quad + m' i_m^2\,\ddot{\vartheta}_{Ej}^{*} - m'l_{ES}\,\ddot{z}_{Ej}^{*}\cos\vartheta_u = 0 \\
&\qquad\qquad\qquad j = 1,2,3\ldots
\end{aligned} \tag{4.16}$$

With Eq. (3.7), this becomes

$$\begin{aligned}
&(EI\,\bar{z}_{Ej}'')'' - (P_{xF}\,e_F\,\bar{\vartheta}_{Ej}\cos\vartheta_u)'' - (P_{xF}\,\bar{z}_{Ej}')' - \nu_j^2 m'\bar{z}_{Ej} + \nu_j^2 m' l_{ES}\,\bar{\vartheta}_{Ej}\cos\vartheta_u = 0 \\
&-[(GJ + P_{xF}\,i_F^2)\,\bar{\vartheta}_{Ej}']' - P_{xF}\,e_F\,\bar{z}_{Ej}''\cos\vartheta_u + \omega_{R0}^2\,m'(i_{m\xi}^2 - i_{m\eta}^2)\,\bar{\vartheta}_{Ej}\cos 2\vartheta_u \\
&\quad - \nu_j^2 m' i_m^2\,\bar{\vartheta}_{Ej} + \nu_j^2 m' l_{ES}\,\bar{z}_{Ej}\cos\vartheta_u = 0 \\
&\qquad\qquad\qquad j = 1,2,3\ldots
\end{aligned} \tag{4.17}$$

For $j = p$ and $j = q$, Eq. (4.17) becomes

$$(EI\bar{z}_{Ep}'')''-(P_{xF}e_F\bar{\vartheta}_{Ep}\cos\vartheta_u)''-(P_{xF}\bar{z}_{Ep}')'-\nu_p^2m'\bar{z}_{Ep}+\nu_p^2m'l_{ES}\bar{\vartheta}_{Ep}\cos\vartheta_u=0$$

$$(EI\bar{z}_{Eq}'')''-(P_{xF}e_F\bar{\vartheta}_{Eq}\cos\vartheta_u)''-(P_{xF}\bar{z}_{Eq}')'-\nu_q^2m'\bar{z}_{Eq}+\nu_q^2m'l_{ES}\bar{\vartheta}_{Eq}\cos\vartheta_u=0$$

$$-\left[(GJ+P_{xF}i_F^2)\bar{\vartheta}_{Ep}'\right]'-P_{xF}e_F\bar{z}_{Ep}''\cos\vartheta_u+\omega_{R0}^2m'(i_{m\zeta}^2-i_{m\eta}^2)\bar{\vartheta}_{Ep}\cos 2\vartheta_u \tag{4.18}$$
$$-\nu_p^2m'i_m^2\bar{\vartheta}_{Ep}+\nu_p^2m'l_{ES}\bar{z}_{Ep}\cos\vartheta_u=0$$

$$-\left[(GJ+P_{xF}i_F^2)\bar{\vartheta}_{Eq}'\right]'-P_{xF}e_F\bar{z}_{Ep}''\cos\vartheta_u+\omega_{R0}^2m'(i_{m\zeta}^2-i_{m\eta}^2)\bar{\vartheta}_{Eq}\cos 2\vartheta_u$$
$$-\nu_p^2m'i_m^2\bar{\vartheta}_{Eq}+\nu_q^2m'l_{ES}\bar{z}_{Eq}\cos\vartheta_u=0$$

We multiply the first equation by $\bar{z}_{Eq}$, the second by $-\bar{z}_{Ep}$, the third by θ_{Eq}, and the fourth by $-\theta_{Ep}$, add these four equations, and integrate over the length of the blade.

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$$\int_0^{R_A}\Big\{(EI\bar{z}_{Ep}'')''\bar{z}_{Eq}-(EI\bar{z}_{Eq}'')''\bar{z}_{Ep}-(P_{xF}e_F\bar{\vartheta}_{Ep}\cos\vartheta_u)''\bar{z}_{Eq}+(P_{xF}e_F\bar{\vartheta}_{Eq}\cos\vartheta_u)''\bar{z}_{Ep}$$
$$-(P_{xF}\bar{z}_{Ep}')'\bar{z}_{Eq}+(P_{xF}\bar{z}_{Eq}')'\bar{z}_{Ep}-\left[(GJ+P_{xF}i_F^2)\bar{\vartheta}_{Ep}'\right]'\bar{\vartheta}_{Eq}+\left[(GJ+P_{xF}i_F^2)\bar{\vartheta}_{Eq}'\right]'\bar{\vartheta}_{Ep}$$
$$-P_{xF}e_F\cos\vartheta_u(\bar{z}_{Ep}''\bar{\vartheta}_{Eq}-\bar{z}_{Eq}''\bar{\vartheta}_{Ep})\Big\}dx= \tag{4.19}$$
$$(\nu_p^2-\nu_q^2)\int_0^{R_A}m'\left[\bar{z}_{Ep}\bar{z}_{Eq}-l_{ES}\cos\vartheta_u(\bar{z}_{Ep}\bar{\vartheta}_{Eq}+\bar{z}_{Eq}\bar{\vartheta}_{Ep})+i_m^2\bar{\vartheta}_{Ep}\bar{\vartheta}_{Eq}\right]dx$$

The terms on the left side are integrated by parts twice, once, or not at all in accordance with Eq. (4.5). We obtain

$$\Big[(EI\bar{z}_{Ep}'')'\bar{z}_{Eq}-(EI\bar{z}_{Eq}'')'\bar{z}_{Ep}-(P_{xF}e_F\bar{\vartheta}_{Ep}\cos\vartheta_u)'\bar{z}_{Eq}+(P_{xF}e_F\bar{\vartheta}_{Eq}\cos\vartheta_u)'\bar{z}_{Ep}$$
$$-EI\bar{z}_{Ep}''\bar{z}_{Eq}'+(EI\bar{z}_{Eq}''\bar{z}_{Ep}'+P_{xF}e_F\bar{\vartheta}_{Ep}\bar{z}_{Eq}'\cos\vartheta_u-P_{xF}e_F\bar{\vartheta}_{Eq}\bar{z}_{Ep}'\cos\vartheta_u \tag{4.20}$$
$$-P_{xF}\bar{z}_{Ep}'\bar{z}_{Eq}+P_{xF}\bar{z}_{Eq}'\bar{z}_{Ep}-(GJ+P_{xF}i_F^2)\bar{\vartheta}_{Ep}'\bar{\vartheta}_{Eq}+(GJ+P_{xF}i_F^2)\bar{\vartheta}_{Eq}'\bar{\vartheta}_{Ep}\Big]_0^{R_A}=$$
$$(\nu_p^2-\nu_q^2)\int_0^{R_A}m'\left[\bar{z}_{Ep}\bar{z}_{Eq}-l_{ES}\cos\vartheta_u(\bar{z}_{Ep}\bar{\vartheta}_{Eq}+\bar{z}_{Eq}\bar{\vartheta}_{Ep})+i_m^2\bar{\vartheta}_{Ep}\bar{\vartheta}_{Eq}\right]dx$$

The integrals on the left side have again disappeared, cf. remark following Eq. (4.13). In this case too, with the aid of the boundary conditions, it can be shown that the expressions on the left side of Eq. (4.20) vanish. Again by Houbolt-Brooks [1] and neglecting the appropriate terms in this case, the bending moment is

$$M_y = EI\, z_E'' - P_{rF} e_F (\sin \vartheta_u + \vartheta_E \cos \vartheta_u) \tag{4.21}$$

At the root of the blade, we may assume $z_E = 0$ and $_E = 0$ as well as $z_E' = 0$ or $M_y = 0$ and $\theta u = 0$, the latter because the axis of the flapping hinge must be the principal axis and the axis of symmetry of the blade cross section at the point concerned. Hence, when $x = 0$, the quantities z_E, θ_E as well as z_E' or EIz_E'', and thus the entire expression in brackets in Eq. (4.20), vanish. The fact that this expression vanishes for $x = R_A$ as well will be proved this time not via moments and shear forces, but using EI, P_{xF} and CJ. Namely, these quantities and all their derivatives vanish at $x = R_A$ (or at $x = R_A + s$ with $s \to + 0$). Eq. (4.20) then becomes /23

$$\boxed{\int_0^{R_A} m' \left[\bar{z}_{Ep} \bar{z}_{Eq} - l_{ES} \cos \vartheta_u (\bar{z}_{Ep} \bar{\vartheta}_{Eq} + \bar{z}_{Eq} \bar{\vartheta}_{Ep}) + i_m^2 \bar{\vartheta}_{Ep} \bar{\vartheta}_{Eq}\right] dx = 0 \quad \text{for} \quad \nu_p \neq \nu_q} \tag{4.22}$$

4.4. Orthogonality Relation for Coupled Flapwise Bending, Edgewise Bending, and Torsion

Now the orthogonality relation for the natural modes defined in Eq. (3.8) will be formulated. With the right sides vanishing, y_{Ej}^*, z_{Ej}^* and θ_{Ej}^* satisfy Eq. (3.4), so

$$\begin{aligned}
&f\,(d, y_{Ej}^*, z_{Ej}^*, \vartheta_{Ej}^*) + m' \ddot{y}_{Ej}^* + m' l_{ES} \ddot{\vartheta}_{Ej}^* \sin \vartheta_u = 0 \\
&g\,(d, y_{Ej}^*, z_{Ej}^*, \vartheta_{Ej}^*) + m' \ddot{z}_{Ej}^* - m' l_{ES} \ddot{\vartheta}_{Ej}^* \cos \vartheta_u = 0 \\
&h\,(d, y_{Ej}^*, z_{Ej}^*, \vartheta_{Ej}^*) + m' i_m^2 \ddot{\vartheta}_{Ej}^* + m' l_{ES} (\ddot{y}_{Ej}^* \sin \vartheta_u - \ddot{z}_{Ej}^* \cos \vartheta_u) = 0
\end{aligned} \tag{4.23}$$

The abbreviations f, g, and h are obtained by comparing Eq. (4.23) with the left side of Eq. (3.4). The symbol d means "differentiation by x." Since f, g, and h are linear in $y_{Ej}*$, $z_{Ej}*$ and $\theta_{Ej}*$, we obtain from Eq. (4.23) with Eq. (3.8)

$$
\begin{aligned}
&f\,(d, \bar{y}_{Ej}, \bar{z}_{Ej}, \bar{\vartheta}_{Ej}) - \nu_j^2 m' \bar{y}_{Ej} - \nu_j^2 m' l_{ES} \bar{\vartheta}_{Ej} \sin\vartheta_u = 0 \\
&g\,(d, \bar{y}_{Ej}, \bar{z}_{Ej}, \bar{\vartheta}_{Ej}) - \nu_j^2 m' \bar{z}_{Ej} + \nu_j^2 m' l_{ES} \bar{\vartheta}_{Ej} \cos\vartheta_u = 0 \qquad (4.24) \\
&h\,(d, \bar{y}_{Ej}, \bar{z}_{Ej}, \bar{\vartheta}_{Ej}) - \nu_j^2 m' i_m^2 \bar{\vartheta}_{Ej} - \nu_j^2 m' l_{ES}(\bar{y}_{Ej}\sin\vartheta_u - \bar{z}_{Ej}\cos\vartheta_u) = 0
\end{aligned}
$$

For j = p and j = q, Eq. (4.24) becomes /24

$$
\begin{aligned}
&f(d, \bar{y}_{Ep}, \bar{z}_{Ep}, \bar{\vartheta}_{Ep}) - \nu_p^2 m' \bar{y}_{Ep} - \nu_p^2 m' l_{ES} \bar{\vartheta}_{Ep} \sin\vartheta_u = 0 \\
&g(d, \bar{y}_{Ep}, \bar{z}_{Ep}, \bar{\vartheta}_{Ep}) - \nu_p^2 m' \bar{z}_{Ep} + \nu_p^2 m' l_{ES} \bar{\vartheta}_{Ep} \cos\vartheta_u = 0 \\
&h(d, \bar{y}_{Ep}, \bar{z}_{Ep}, \bar{\vartheta}_{Ep}) - \nu_p^2 m' i_m^2 \bar{\vartheta}_{Ep} - \nu_p^2 m' l_{ES}(\bar{y}_{Ep}\sin\vartheta_u - \bar{z}_{Ep}\cos\vartheta_u) = 0 \\
&f(d, \bar{y}_{Eq}, \bar{z}_{Eq}, \bar{\vartheta}_{Eq}) - \nu_q^2 m' \bar{y}_{Eq} - \nu_q^2 m' l_{ES} \bar{\vartheta}_{Eq} \sin\vartheta_u = 0 \qquad (4.25) \\
&g(d, \bar{y}_{Eq}, \bar{z}_{Eq}, \bar{\vartheta}_{Eq}) - \nu_q^2 m' \bar{z}_{Eq} + \nu_q^2 m' l_{ES} \bar{\vartheta}_{Eq} \cos\vartheta_u = 0 \\
&h(d, \bar{y}_{Eq}, \bar{z}_{Eq}, \bar{\vartheta}_{Eq}) - \nu_q^2 m' i_m^2 \bar{\vartheta}_{Eq} - \nu_q^2 m' l_{ES}(\bar{y}_{Eq}\sin\vartheta_u - \bar{z}_{Eq}\cos\vartheta_u) = 0
\end{aligned}
$$

In this case, we multiply through by $\bar{y}_{Eq}$, $\bar{z}_{Eq}$, $\bar{\theta}_{Eq}$, $-\bar{y}_{Ep}$, $-\bar{z}_{Ep}$, and $-\bar{\theta}_{Ep}$, then add all six equations, integrate over the length of the blade, and obtain

$$
\begin{aligned}
&\int_0^{R_A} \Big[\bar{y}_{Eq}\cdot f(d, \bar{y}_{Ep}, \bar{z}_{Ep}, \bar{\vartheta}_{Ep}) - \bar{y}_{Ep}\cdot f(d, \bar{y}_{Eq}, \bar{z}_{Eq}, \bar{\vartheta}_{Eq}) \\
&+ \bar{z}_{Eq}\cdot g(d, \bar{y}_{Ep}, \bar{z}_{Ep}, \bar{\vartheta}_{Ep}) - \bar{z}_{Ep}\cdot g(d, \bar{y}_{Eq}, \bar{z}_{Eq}, \bar{\vartheta}_{Eq}) \\
&+ \bar{\vartheta}_{Eq}\cdot h(d, \bar{y}_{Ep}, \bar{z}_{Ep}, \bar{\vartheta}_{Ep}) - \bar{\vartheta}_{Ep}\cdot h(d, \bar{y}_{Eq}, \bar{z}_{Eq}, \bar{\vartheta}_{Eq})\Big]\,dx \qquad (4.26) \\
&= (\nu_p^2 - \nu_q^2)\cdot \int_0^{R_A} m' \Big[\bar{y}_{Ep}\,\bar{y}_{Eq} + \bar{z}_{Ep}\,\bar{z}_{Eq} + i_m^2\,\bar{\vartheta}_{Ep}\,\bar{\vartheta}_{Eq} \\
&+ l_{ES}(\bar{y}_{Ep}\bar{\vartheta}_{Eq} + \bar{y}_{Eq}\bar{\vartheta}_{Ep})\sin\vartheta_u - l_{ES}(\bar{z}_{Ep}\bar{\vartheta}_{Eq} + \bar{z}_{Eq}\bar{\vartheta}_{Ep})\cos\vartheta_u\Big]\,dx
\end{aligned}
$$

Writing out the functions f, g, and h explicity, Eq. (4.26) becomes

$$\int_0^{R_A}\left[EI_c\bar{y}_{Ep}''+EI_0\bar{z}_{Ep}''+P_{xF}e_F\bar{\vartheta}_{Ep}\sin\vartheta_u-EB_2\vartheta_u'\bar{\vartheta}_{Ep}'\cos\vartheta_u\right]''\bar{y}_{Eq}\,dx$$ /25

$$-\int_0^{R_A}\left[EI_c\bar{y}_{Eq}''+EI_0\bar{z}_{Eq}''+P_{xF}e_F\bar{\vartheta}_{Eq}\sin\vartheta_u-EB_2\vartheta_u'\bar{\vartheta}_{Eq}'\cos\vartheta_u\right]''\bar{y}_{Ep}\,dx$$

$$+\int_0^{R_A}\left[EI_0\bar{y}_{Ep}''+EI_s\bar{z}_{Ep}''-P_{xF}e_F\bar{\vartheta}_{Ep}\cos\vartheta_u-EB_2\vartheta_u'\bar{\vartheta}_{Ep}'\sin\vartheta_u\right]''\bar{z}_{Eq}\,dx$$

$$-\int_0^{R_A}\left[EI_0\bar{y}_{Eq}''+EI_s\bar{z}_{Eq}''-P_{xF}e_F\bar{\vartheta}_{Eq}\cos\vartheta_u-EB_2\vartheta_u'\bar{\vartheta}_{Eq}'\sin\vartheta_u\right]''\bar{z}_{Ep}\,dx$$

$$-\int_0^{R_A}\left[(GJ+P_{xF}i_F^2+EB_1\vartheta_u'^2)\bar{\vartheta}_{Ep}'-EB_2\vartheta_u'(\bar{y}_{Ep}''\cos\vartheta_u+\bar{z}_{Ep}''\sin\vartheta_u)\right]'\bar{\vartheta}_{Eq}\,dx$$

$$+\int_0^{R_A}\left[(GJ+P_{xF}i_F^2+EB_1\vartheta_u'^2)\bar{\vartheta}_{Eq}'-EB_2\vartheta_u'(\bar{y}_{Eq}''\cos\vartheta_u+\bar{z}_{Eq}''\sin\vartheta_u)\right]'\bar{\vartheta}_{Ep}\,dx$$

$$-\int_0^{R_A}\left\{(P_{xF}\bar{y}_{Ep}')'+\omega_{R0}^2\left[m'l_{ES}(x+a)\bar{\vartheta}_{Ep}\sin\vartheta_u\right]'+\underline{\omega_{R0}^2m'l_{ES}\bar{\vartheta}_{Ep}\sin\vartheta_u}+\underline{\omega_{R0}^2m'\bar{y}_{Ep}}\right\}\bar{y}_{Eq}\,dx \qquad (4.27)$$

$$+\int_0^{R_A}\left\{(P_{xF}\bar{y}_{Eq}')'+\omega_{R0}^2\left[m'l_{ES}(x+a)\bar{\vartheta}_{Eq}\sin\vartheta_u\right]'+\underline{\omega_{R0}^2m'l_{ES}\bar{\vartheta}_{Eq}\sin\vartheta_u}+\underline{\omega_{R0}^2m'\bar{y}_{Eq}}\right\}\bar{y}_{Ep}\,dx$$

$$+\int_0^{R_A}\left\{-(P_{xF}\bar{z}_{Ep}')'+\omega_{R0}^2\left[m'l_{ES}(x+a)\bar{\vartheta}_{Ep}\cos\vartheta_u\right]'\right\}\bar{z}_{Eq}\,dx$$

$$-\int_0^{R_A}\left\{-(P_{xF}\bar{z}_{Eq}')'+\omega_{R0}^2\left[m'l_{ES}(x+a)\bar{\vartheta}_{Eq}\cos\vartheta_u\right]'\right\}\bar{z}_{Ep}\,dx$$

$$-\int_0^{R_A}\left[P_{xF}e_F(\bar{z}_{Ep}''\cos\vartheta_u-\bar{y}_{Ep}''\sin\vartheta_u)+\omega_{R0}^2m'l_{ES}(x+a)(\bar{z}_{Ep}'\cos\vartheta_u-\bar{y}_{Ep}'\sin\vartheta_u)+\underline{\omega_{R0}^2m'l_{ES}\bar{y}_{Ep}\sin\vartheta_u}\right]\bar{\vartheta}_{Eq}\,dx$$

$$+\int_0^{R_A}\left[P_{xF}e_F(\bar{z}_{Eq}''\cos\vartheta_u-\bar{y}_{Eq}''\sin\vartheta_u)+\omega_{R0}^2m'l_{ES}(x+a)(\bar{z}_{Eq}'\cos\vartheta_u-\bar{y}_{Eq}'\sin\vartheta_u)+\underline{\omega_{R0}^2m'l_{ES}\bar{y}_{Eq}\sin\vartheta_u}\right]\bar{\vartheta}_{Ep}\,dx$$

= right side of Eq. (4.26).

In the last two brackets, one expression was omitted in each case, since these two expressions obviously cancel each other. Continuing, the underlined expressions also cancel each other out. Of the remaining terms, those in lines 1 through 4 are integrated by parts twice, and those in lines 5 through 10 once, in accordance with Eq. (4.5). This eliminates all the terms preceded by integral signs, and we obtain: /26

$$
\begin{aligned}
&\Big\{\left[EI_c\bar{y}_{Ep}''+EI_0\bar{z}_{Ep}''+P_{xF}e_F\bar{\vartheta}_{Ep}\sin\vartheta_u-EB_2\vartheta_u'\bar{\vartheta}_{Ep}'\cos\vartheta_u\right]'\bar{y}_{Eq}\\
&-\left[EI_c\bar{y}_{Eq}''+EI_0\bar{z}_{Eq}''+P_{xF}e_F\bar{\vartheta}_{Eq}\sin\vartheta_u-EB_2\vartheta_u'\bar{\vartheta}_{Eq}'\cos\vartheta_u\right]'\bar{y}_{Ep}\\
&+\left[EI_0\bar{y}_{Ep}''+EI_s\bar{z}_{Ep}''-P_{xF}e_F\bar{\vartheta}_{Ep}\cos\vartheta_u-EB_2\vartheta_u'\bar{\vartheta}_{Ep}'\sin\vartheta_u\right]'\bar{z}_{Eq}\\
&-\left[EI_0\bar{y}_{Eq}''+EI_s\bar{z}_{Eq}''-P_{xF}e_F\bar{\vartheta}_{Eq}\cos\vartheta_u-EB_2\vartheta_u'\bar{\vartheta}_{Eq}'\sin\vartheta_u\right]'\bar{z}_{Ep}\\
&-\left[EI_c\bar{y}_{Ep}''+EI_0\bar{z}_{Ep}''+P_{xF}e_F\bar{\vartheta}_{Ep}\sin\vartheta_u-EB_2\vartheta_u'\bar{\vartheta}_{Ep}'\cos\vartheta_u\right]\bar{y}_{Eq}'\\
&+\left[EI_c\bar{y}_{Eq}''+EI_0\bar{z}_{Eq}''+P_{xF}e_F\bar{\vartheta}_{Eq}\sin\vartheta_u-EB_2\vartheta_u'\bar{\vartheta}_{Eq}'\cos\vartheta_u\right]\bar{y}_{Ep}'\\
&-\left[EI_0\bar{y}_{Ep}''+EI_s\bar{z}_{Ep}''-P_{xF}e_F\bar{\vartheta}_{Ep}\cos\vartheta_u-EB_2\vartheta_u'\bar{\vartheta}_{Ep}'\sin\vartheta_u\right]\bar{z}_{Eq}'\\
&+\left[EI_0\bar{y}_{Eq}''+EI_s\bar{z}_{Eq}''-P_{xF}e_F\bar{\vartheta}_{Eq}\cos\vartheta_u-EB_2\vartheta_u'\bar{\vartheta}_{Eq}'\sin\vartheta_u\right]\bar{z}_{Ep}'\\
&-\left[\left(GJ+P_{xF}i_F^2+EB_1\vartheta_u'^2\right)\bar{\vartheta}_{Ep}'-EB_2\vartheta_u'\left(\bar{y}_{Ep}''\cos\vartheta_u+\bar{z}_{Ep}''\sin\vartheta_u\right)\right]\bar{\vartheta}_{Eq}\\
&+\left[\left(GJ+P_{xF}i_F^2+EB_1\vartheta_u'^2\right)\bar{\vartheta}_{Eq}'-EB_2\vartheta_u'\left(\bar{y}_{Eq}''\cos\vartheta_u+\bar{z}_{Eq}''\sin\vartheta_u\right)\right]\bar{\vartheta}_{Ep}\\
&-\left[P_{xF}\bar{y}_{Ep}'+\omega_{Ro}^2m'l_{ES}(x+a)\bar{\vartheta}_{Ep}'\sin\vartheta_u\right]\bar{y}_{Eq}+\left[P_{xF}\bar{y}_{Eq}'+\omega_{Ro}^2m'l_{ES}(x+a)\bar{\vartheta}_{Eq}\sin\vartheta_u\right]\bar{y}_{Ep}\\
&-\left[P_{xF}\bar{z}_{Ep}'-\omega_{Ro}^2m'l_{ES}(x+a)\bar{\vartheta}_{Ep}'\cos\vartheta_u\right]\bar{z}_{Eq}+\left[P_{xF}\bar{z}_{Eq}'+\omega_{Ro}^2m'l_{ES}(x+a)\bar{\vartheta}_{Eq}\cos\vartheta_u\right]\bar{z}_{Ep}\Big\}_0^{\ell_A}
\end{aligned}
\qquad (4.28)
$$

= right side of Eq. (4.26)

It is highly reasonable to assume that the left side of Eq. (4.28) will disappear just like those in Eqs. (4.6), (4.13), and (4.20). Since the expression in wavy brackets vanishes at $x = 0$, we obtain from $y_E = z_E = \theta_E = y_E' = z_E' = 0$ for $x = 0$ in the case of rigid restraint at the ends. For a flapping or swivel hinge, it will be -- instead of z_E' and y_E' -- the corresponding expressions in brackets in line 5 through 8 of Eq. (4.28), the ones multiplied by z_E' or y_E', which are equal to zero. According to Houbolt and Brooks [1], these square-bracketed expressions constitute the bending moments about the y and z axes, apart from a constant term $P_{xF}\, e_F \sin\theta_u$ or $P_{xF}\, e_F \cos\theta_u$. /27

However, in the case of the flapping hinge, we may assume $\theta_u = 0$ for $x = 0$ by the remark following Eq. (4.21), and in the case of the swivel hinge, $e_F = 0$ for $x = 0$.

At the tip of the blade, the quantities EI_c, EI_o, EI_s, GJ, P_{xF}, EB_1, EB_2, and m' vanish along with all their derivatives. The expression in wavy brackets therefore vanishes for $x = R_A$ as well, so that the left side of Eq. (4.28) turns out to be identically zero. The desired orthogonality condition then reads

$$\boxed{\begin{aligned}&\int_0^{R_A} m'\Big[\bar{y}_{Ep}\bar{y}_{Eq} + \bar{z}_{Ep}\bar{z}_{Eq} + i_m^2\bar{\vartheta}_{Ep}\bar{\vartheta}_{Eq}\\ &+ l_{ES}(\bar{y}_{Ep}\bar{\vartheta}_{Eq} + \bar{y}_{Eq}\bar{\vartheta}_{Ep})\sin\vartheta_u - l_{ES}(\bar{z}_{Ep}\bar{\vartheta}_{Eq} + \bar{z}_{Eq}\bar{\vartheta}_{Ep})\cos\vartheta_u\Big]dx = 0\\ &\text{for}\quad \nu_p \neq \nu_q\end{aligned}} \tag{4.29}$$

Perhaps the orthogonality conditions can be derived in a more general and elegant fashion from (virtual) work principles, e.g. by the Ritz method. The boundary conditions would then have the simple and general form "boundary work = zero."

5. Methods of Solution /28

5.1. Method of Solution for Uncoupled Flapwise Bending

We must solve Eq. (3.1):

$$(EIz_E'')'' - (P_{xF}z_E')' + m'\ddot{z}_E = Z'(x, t, \dot{z}_E) \tag{5.1}$$

For this purpose, we use the trial solution discussed in Chapter 1:

$$z_E(x, t) = \sum_{j=1}^{n} \bar{z}_{Ej}(x)\cdot q_j(t) \tag{5.2}$$

$\bar{z}_{Ej}$ are the (known) natural modes, and q_j the unknown functions of time. Using Eq. (5.2), we obtain from Eq. (5.1):

$$\sum_{j=1}^{n}\left[(EI\bar{z}_{Ej}'')''q_j-(P_{xF}\bar{z}_{Ej}')'q_j+m'\bar{z}_{Ej}\ddot{q}_j\right]=Z'(x,t,\dot{z}_E) \qquad (5.3)$$

Now we use Eq. (4.2) for the natural modes and frequencies.

$$(EI\bar{z}_{Ej}'')''-(P_{xF}\bar{z}_{Ej}')'-\nu_j^2 m'\bar{z}_{Ej}=0 \qquad j=1,2,3\ldots \qquad (5.4)$$

We multiply by q_j and sum from $j = 1$ to $j = n$:

$$\sum_{j=1}^{n}\left[(EI\bar{z}_{Ej}'')''q_j-(P_{xF}\bar{z}_{Ej}')'q_j-\nu_j^2 m'\bar{z}_{Ej}q_j\right]=0 \qquad (5.5)$$

Subtracting Eq. (5.5) from Eq. (5.3), we obtain

$$\sum_{j=1}^{n}m'\bar{z}_{Ej}(\ddot{q}_j+\nu_j^2 q_j)=Z'(x,t,\dot{z}_E) \qquad (5.6)$$

We multiply by $\bar{z}_{Ei}$ and integrate over the blade

$$\sum_{j=1}^{n}(\ddot{q}_j+\nu_j^2 q_j)\int_0^{R_A}m'\bar{z}_{Ei}\bar{z}_{Ej}\,dx=\int_0^{R_A}Z'(x,t,\dot{z}_E)\bar{z}_{Ei}\,dx \qquad (5.7)$$

Because of the orthogonality condition (4.8), all terms in the sum on the left side of Eq. (5.7) vanish except for the one with $j = i$. Hence, Eq. (5.7) becomes

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$$\ddot{q}_i+\nu_i^2 q_i=\frac{\int_0^{R_A}Z'(x,t,\dot{z}_E)\bar{z}_{Ei}\,dx}{\int_0^{R_A}m'\bar{z}_{Ei}^2\,dx} \qquad (5.8)$$

This equation has already appeared in DFVLR Report 98 by Just and Storm [2]. Applying it for $i = 1$ through n, we obtain n differential equations to calculate the n functions $q_i(t)$ -- in Eq. (5.2), $q_j(t)$. The differential equations are coupled to one another via Z', in which z_E is to be expressed in terms of the q_j by Eq. (5.2). In general, these equations can be solved only numerically, particularly if the c_a-α curve is nonlinear.

If the c_a-α curve is linear, Z' has a form which is of the degree of difficulty of Eq. (5.9). In this case, an analytic solution is conceivable, and we will discuss this topic briefly.

$$Z'(x, t, \dot{z}_E) = a(x)\sin\omega_{Ro}t + b(x)\cdot(1+\lambda\sin\omega_{Ro}t)\,\dot{z}_E \tag{5.9}$$

Hence, the rotor blade is acted on by an excitation term of the usual type, and also by damping which is a function of the blade-rotation angle and $\dot{z}_E$. This damping has been mentioned in the work of H. Schmidt [3], p. 120, Eq. (15+1). Eqs. (5.8) and (5.9) imply

$$\ddot{q}_i + \nu_i^2 q_i = \frac{\int_0^{R_A} a(x)\bar{z}_{Ei}\,dx}{\int_0^{R_A} m'\bar{z}_{Ei}^2\,dx}\sin\omega_{Ro}t + \frac{\int_0^{R_A} b(x)\dot{z}_E\bar{z}_{Ei}\,dx}{\int_0^{R_A} m'\bar{z}_{Ei}^2\,dx}(1+\lambda\sin\omega_{Ro}t) \tag{5.10}$$

For $\dot{z}_E$, we use Eq. (5.2), and obtain

$$\ddot{q}_i + \nu_i^2 q_i = \alpha_i\sin\omega_{Ro}t + (1+\lambda\sin\omega_{Ro}t)\sum_{j=1}^{n}\dot{q}_j\frac{\int_0^{R_A} b(x)\bar{z}_{Ej}\bar{z}_{Ei}\,dx}{\int_0^{R_A} m'\bar{z}_{Ei}^2\,dx} \tag{5.11}$$

or /30

$$\ddot{q}_i - (1+\lambda\sin\omega_{Ro}t)\sum_{j=1}^{n}\beta_{ij}\dot{q}_j + \nu_i^2 q_i = \alpha_i\sin\omega_{Ro}t \tag{5.12}$$

with

$$\alpha_i = \frac{\int_0^{R_A} a(x)\bar{z}_{Ei}\,dx}{\int_0^{R_A} m'\bar{z}_{Ei}^2\,dx} \qquad \beta_{ij} = \frac{\int_0^{R_A} b(x)\bar{z}_{Ei}\bar{z}_{Ej}\,dx}{\int_0^{R_A} m'\bar{z}_{Ei}^2\,dx} \tag{5.13}$$

Eq. (5.12) can be written out for $i = 1, 2, 3, \ldots, n$. The resulting system of coupled ordinary differential equations has time-dependent coefficients for the q_i. It can be solved stepwise, e.g. by the Runge-Kutta method, or also by using harmonic trial solutions for the q_i. In that case, the result is $m \cdot n$ algebraic equations with $m \cdot n$ unknowns, where m is the number of harmonics taken into account, and n the number of natural modes taken into account. The constants α_i and β_{ij} can easily be calculated by replacing the integrals with sums and substituting in the discrete values calculated for the natural modes by the segment method.

5.2. Method of Solution for Coupled Flapwise and Edgewise Bending

The equation to be solved in (3.2), which states

$$\begin{aligned}
&(EI_C y_E'')'' + (EI_0 z_E'')'' - (P_{xF} y_E')' + m'\ddot{y}_E - \omega_{R0}^2 m' y_E = Y'(x,t,\dot{z}_E) \\
&(EI_0 y_E'')'' + (EI_S z_E'')'' - (P_{xF} z_E')' + m'\ddot{z}_E = Z'(x,t,\dot{z}_E)
\end{aligned} \tag{5.14}$$

For this purpose, we choose a trial solution in the form of the coupled natural modes

$$\begin{bmatrix} y_E \\ z_E \end{bmatrix}(x,t) = \sum_{j=1}^{n} \begin{bmatrix} \bar{y}_E \\ \bar{z}_E \end{bmatrix}_j (x) \cdot q_j(t) \tag{5.15}$$

Using Eq. (5.15), we obtain from Eq. (5.14): /31

$$\begin{aligned}
&\sum_{j=1}^{n} \left[(EI_C \bar{y}_{Ej}'')'' q_j + (EI_0 \bar{z}_{Ej}'')'' q_j - (P_{xF} \bar{y}_{Ej}')' q_j + m' \bar{y}_{Ej} \ddot{q}_j - \omega_{R0}^2 m' \bar{y}_{Ej} q_j \right] = Y'(x,t,\dot{z}_E) \\
&\sum_{j=1}^{n} \left[(EI_0 \bar{y}_{Ej}'')'' q_j + (EI_S \bar{z}_{Ej}'')'' q_j - (P_{xF} \bar{z}_{Ej}')' q_j + m' \bar{z}_{Ej} \ddot{q}_j \right] = Z'(x,t,\dot{z}_E)
\end{aligned} \tag{5.16}$$

Eq. (4.10) states

$$(EI_c \bar{y}_{Ej}'')'' + (EI_0 \bar{z}_{Ej}'')'' - (P_{xF} \bar{y}_{Ej}')' - (\nu_j^2 + \omega_{R0}^2) m' \bar{y}_{Ej} = 0$$

$$(EI_0 \bar{y}_{Ej}'')'' + (EI_S \bar{z}_{Ej}'')'' - (P_{xF} \bar{z}_{Ej}')' - \nu_j^2 m' \bar{z}_{Ej} \qquad = 0 \tag{5.17}$$

We multiply by q_j and sum from $j = 1$ to $j = n$

$$\sum_{j=1}^{n} \left[(EI_c \bar{y}_{Ej}'')'' q_j + (EI_0 \bar{z}_{Ej}'')'' q_j - (P_{xF} \bar{y}_{Ej}')' q_j - (\nu_j^2 + \omega_{R0}^2) m' \bar{y}_{Ej} q_j \right] = 0$$

$$\sum_{j=1}^{n} \left[(EI_0 \bar{y}_{Ej}'')'' q_j + (EI_S \bar{z}_{Ej}'')'' q_j - (P_{xF} \bar{z}_{Ej}')' q_j - \nu_j^2 m' \bar{z}_{Ej} q_j \right] \quad = 0 \tag{5.18}$$

If Eq. (5.18) is subtracted from (5.16), the remainder is

$$\sum_{j=1}^{n} m' \bar{y}_{Ej} (\ddot{q}_j + \nu_j^2 q_j) = Y'(x, t, \dot{z}_E)$$

$$\sum_{j=1}^{n} m' \bar{z}_{Ej} (\ddot{q}_j + \nu_j^2 q_j) = Z'(x, t, \dot{z}_E) \tag{5.19}$$

We multiply through by $\bar{y}_{Ei}$ or $\bar{z}_{Ei}$, integrate, and add:

$$\sum_{j=1}^{n} (\ddot{q}_j + \nu_j^2 q_j) \int_0^{\ell_A} m' (\bar{y}_{Ei} \bar{y}_{Ej} + \bar{z}_{Ei} \bar{z}_{Ej}) dx = \int_0^{\ell_A} \left[Y'(x, t, \dot{z}_E) \bar{y}_{Ei} + Z'(x, t, \dot{z}_E) \bar{z}_{Ei} \right] dx \tag{5.20}$$

Because of the orthogonality condition (4.15), all of the terms in the sum on the left side vanish except for the one with $j = i$. Hence, Eq. (5.20) becomes /32

$$\ddot{q}_i + \nu_i^2 q_i = \frac{\int_0^{\ell_A} \left[Y'(x, t, \dot{z}_E) \bar{y}_{Ei} + Z'(x, t, \dot{z}_E) \bar{z}_{Ei} \right] dx}{\int_0^{\ell_A} m' (\bar{y}_{Ei}^2 + \bar{z}_{Ei}^2) dx} \tag{5.21}$$

5.3. Method of Solution for Coupled Flapwise Bending and Torsion

The equation to be solved in (3.3). It states

$$\begin{aligned}&(EIz_E'')''-(P_{xF}e_F\vartheta_E\cos\vartheta_u)''-(P_{xF}z_E')'+m'\ddot{z}_E-m'l_{ES}\ddot{\vartheta}_E\cos\vartheta_u=Z'(x,t,\dot{z}_E,\vartheta_E,\dot{\vartheta}_E)\\&-\left[(GJ+P_{xF}i_F^2)\vartheta_E'\right]'-P_{xF}e_F z_E''\cos\vartheta_u+\omega_{Ro}^2m'(i_{m\xi}^2-i_{m\eta}^2)\vartheta_E\cos2\vartheta_u\\&+m'i_m^2\ddot{\vartheta}_E-m'l_{ES}\ddot{z}_E\cos\vartheta_u=M'(x,t,\dot{z}_E,\vartheta_E,\dot{\vartheta}_E)\end{aligned}\tag{5.22}$$

The trial solution in the form of coupled natural modes reads

$$\begin{bmatrix}z_E\\ \vartheta_E\end{bmatrix}(x,t)=\sum_{j=1}^{n}\begin{bmatrix}\bar{z}_E\\ \bar{\vartheta}_E\end{bmatrix}_j(x)\cdot q_j(t)\tag{5.23}$$

Using Eq. (5.23), we obtain from Eq. (5.22):

$$\begin{aligned}&\sum_{j=1}^{n}\left[(EI\bar{z}_{Ej}'')''q_j-(P_{xF}e_F\bar{\vartheta}_{Ej}\cos\vartheta_u)''q_j-(P_{xF}\bar{z}_{Ej}')'q_j+m'\bar{z}_{Ej}\ddot{q}_j-m'l_{ES}\bar{\vartheta}_{Ej}\ddot{q}_j\cos\vartheta_u\right]\\&\qquad=Z'(x,t,\dot{z}_E,\vartheta_E,\dot{\vartheta}_E)\\&\sum_{j=1}^{n}\Big\{-\left[(GJ+P_{xF}i_F^2)\bar{\vartheta}_{Ej}'\right]'q_j-P_{xF}e_F\bar{z}_{Ej}''q_j\cos\vartheta_u+\omega_{Ro}^2m'(i_{m\xi}^2-i_{m\eta}^2)\bar{\vartheta}_{Ej}q_j\cos2\vartheta_u\\&\qquad+m'i_m^2\bar{\vartheta}_{Ej}\ddot{q}_j-m'l_{ES}\bar{z}_{Ej}\ddot{q}_j\cos\vartheta_u\Big\}=M'(x,t,\dot{z}_E,\vartheta_E,\dot{\vartheta}_E)\end{aligned}\tag{5.24}$$

Eq. (4.17) states /33

$$\begin{aligned}&(EI\bar{z}_{Ej}'')''-(P_{xF}e_F\bar{\vartheta}_{Ej}\cos\vartheta_u)''-(P_{xF}\bar{z}_{Ej}')'-\nu_j^2m'\bar{z}_{Ej}+\nu_j^2m'l_{ES}\bar{\vartheta}_{Ej}\cos\vartheta_u=0\\&-\left[(GJ+P_{xF}i_F^2)\bar{\vartheta}_{Ej}'\right]'-P_{xF}e_F\bar{z}_{Ej}''\cos\vartheta_u+\omega_{Ro}^2m'(i_{m\xi}^2-i_{m\eta}^2)\bar{\vartheta}_{Ej}\cos2\vartheta_u\\&\qquad-\nu_j^2m'i_m^2\bar{\vartheta}_{Ej}+\nu_j^2m'l_{ES}\bar{z}_{Ej}\cos\vartheta_u=0\end{aligned}\tag{5.25}$$

We multiply by q_j and sum from $j = 1$ to $j = n$.

$$\sum_{j=1}^{n} \left[(EI\bar{z}_{Ej}'')'' q_j - (P_{xF} e_F \bar{\vartheta}_{Ej} \cos\vartheta_U)'' q_j - (P_{xF} \bar{z}_{Ej}')' q_j - \nu_j^2 m' \bar{z}_{Ej} q_j + \nu_j^2 m' l_{ES} \bar{\vartheta}_{Ej} q_j \cos\vartheta_U \right] = 0$$

$$\begin{aligned} \sum_{j=1}^{n} \Big\{ -\left[(GJ + P_{xF} i_F^2) \bar{\vartheta}_{Ej}' \right]' q_j - P_{xF} e_F \bar{z}_{Ej}'' q_j \cos\vartheta_U + \omega_{R0}^2 m' (i_{m\xi}^2 - i_{m\eta}^2) \bar{\vartheta}_{Ej} q_j \cos 2\vartheta_U \\ - \nu_j^2 m' i_m^2 \bar{\vartheta}_{Ej} q_j + \nu_j^2 m' l_{ES} \bar{z}_{Ej} q_j \cos\vartheta_U \Big\} = 0 \end{aligned} \tag{5.26}$$

If Eq. (5.26) is subtracted from Eq. (5.24), the remainder is

$$\begin{aligned} \sum_{j=1}^{n} m' (\bar{z}_{Ej} - \bar{\vartheta}_{Ej} l_{ES} \cos\vartheta_U)(\ddot{q}_j + \nu_j^2 q_j) &= Z'(x, t, \dot{z}_E, \vartheta_E, \dot{\vartheta}_E) \\ \sum_{j=1}^{n} m' (\bar{\vartheta}_{Ej} i_m^2 - \bar{z}_{Ej} l_{ES} \cos\vartheta_U)(\ddot{q}_j + \nu_j^2 q_j) &= M'(x, t, \dot{z}_E, \vartheta_E, \dot{\vartheta}_E) \end{aligned} \tag{5.27}$$

We multiply through by $\bar{z}_{Ei}$ or $\bar{\theta}_{Ei}$, integrate, and add:

$$\begin{aligned} \sum_{j=1}^{n} (\ddot{q}_j + \nu_j^2 q_j) \cdot \int_0^{R_A} m' \left[\bar{z}_{Ei} \bar{z}_{Ej} - l_{ES} \cos\vartheta_U (\bar{z}_{Ei} \bar{\vartheta}_{Ej} + \bar{z}_{Ej} \bar{\vartheta}_{Ei}) + i_m^2 \bar{\vartheta}_{Ei} \bar{\vartheta}_{Ej} \right] dx \\ = \int_0^{R_A} \left[Z'(x, t, \dot{z}_E, \vartheta_E, \dot{\vartheta}_E) \cdot \bar{z}_{Ei} + M'(x, t, \dot{z}_E, \vartheta_E, \dot{\vartheta}_E) \cdot \bar{\vartheta}_{Ei} \right] dx \end{aligned} \tag{5.28}$$

Because of the orthogonality condition (4.22), all terms in sum on the left side vanish except for the one with $j = i$. Therefore, we write: /34

$$\ddot{q}_i + \nu_i^2 q_i = \frac{\int_0^{R_A} \left[Z'(x, t, \dot{z}_E, \vartheta_E, \dot{\vartheta}_E) \cdot \bar{z}_{Ei} + M'(x, t, \dot{z}_E, \vartheta_E, \dot{\vartheta}_E) \cdot \bar{\vartheta}_{Ei} \right] dx}{\int_0^{R_A} m' \left[\bar{z}_{Ei}^2 - 2 l_{ES} \cos\vartheta_U \bar{z}_{Ei} \bar{\vartheta}_{Ei} + i_m^2 \bar{\vartheta}_{Ei}^2 \right] dx} \tag{5.29}$$

5.4. Method of Solution for Coupled Flapwise Bending, Edgewise Bending, and Torsion

The equation to be solved is (3.4). With the abbreviations from Chapter 4, it states

$$
\begin{aligned}
f(d, y_E, z_E, \vartheta_E) + m' \ddot{y}_E + m' l_{ES} \ddot{\vartheta}_E \sin\vartheta_u &= Y_e'(x, t, \dot{y}_E, \dot{z}_E, \vartheta_E) \\
g(d, y_E, z_E, \vartheta_E) + m' \ddot{z}_E - m' l_{ES} \ddot{\vartheta}_E \cos\vartheta_u &= Z_e'(x, t, \dot{y}_E, \dot{z}_E, \vartheta_E, \dot{\vartheta}_E) \qquad (5.30) \\
h(d, y_E, z_E, \vartheta_E) + m' i_m^2 \ddot{\vartheta}_E + m' l_{ES} (\ddot{y}_E \sin\vartheta_u - \ddot{z}_E \cos\vartheta_u) &= M_e'(x, t, \dot{y}_E, \dot{z}_E, \vartheta_E, \dot{\vartheta}_E)
\end{aligned}
$$

At this point, we introduce three other abbreviations:

$$
\begin{aligned}
Y_e'(x, t, \dot{y}_E, \dot{z}_E, \vartheta_E) &= Y'(x, t, \dot{y}_E, \dot{z}_E, \vartheta_E) + (P_{xF} e_F \cos\vartheta_u)' \\
&\quad - \omega_{Ro}^2 \left[m' l_{ES} (x+a) \cos\vartheta_u \right]' + \omega_{Ro}^2 m' (e_A - l_{ES} \cos\vartheta_u) \\
Z_e'(x, t, \dot{y}_E, \dot{z}_E, \vartheta_E, \dot{\vartheta}_E) &= Z'(x, t, \dot{y}_E, \dot{z}_E, \vartheta_E, \dot{\vartheta}_E) + (P_{xF} e_F \sin\vartheta_u)' - \omega_{Ro}^2 \left[m' l_{ES} (x+a) \sin\vartheta_u \right]' \qquad (5.31) \\
M_e'(x, t, \dot{y}_E, \dot{z}_E, \vartheta_E, \dot{\vartheta}_E) &= M'(x, t, \dot{y}_E, \dot{z}_E, \vartheta_E, \dot{\vartheta}_E) + (P_{xF} i_F^2 \vartheta_u')' \\
&\quad - \omega_{Ro}^2 \, m' \left[(i_{m\zeta}^2 - i_{m\eta}^2) \sin\vartheta_u \cos\vartheta_u - l_{ES} e_A \sin\vartheta_u \right]
\end{aligned}
$$

Again, we choose a trial solution with coupled natural modes: /35

$$
\begin{bmatrix} y_E \\ z_E \\ \vartheta_E \end{bmatrix} (x, t) = \sum_{j=1}^{n} \begin{bmatrix} \bar{y}_E \\ \bar{z}_E \\ \bar{\vartheta}_E \end{bmatrix}_j (x) \cdot q_j(t) \qquad (5.32)
$$

Since f, g, and h are linear functions of the unknowns, Eqs. (5.30) and (5.32) yield:

$$\sum_{j=1}^{n}\left[f(d,\bar{y}_{Ej},\bar{z}_{Ej},\bar{\vartheta}_{Ej})\cdot q_j+m'\bar{y}_{Ej}\cdot\ddot{q}_j+m'l_{ES}\bar{\vartheta}_{Ej}\ddot{q}_j\sin\vartheta_u\right] = Y_e'(x,t,\dot{y}_E,\dot{z}_E,\vartheta_E)$$

$$\sum_{j=1}^{n}\left[g(d,\bar{y}_{Ej},\bar{z}_{Ej},\vartheta_{Ej})\cdot q_j+m'\bar{z}_{Ej}\cdot\ddot{q}_j-m'l_{ES}\bar{\vartheta}_{Ej}\ddot{q}_j\cos\vartheta_u\right] = Z_e'(x,t,\dot{y}_E,\dot{z}_E,\vartheta_E,\dot{\vartheta}_E) \qquad (5.33)$$

$$\sum_{j=1}^{n}\left[h(d,\bar{y}_{Ej},\bar{z}_{Ej},\bar{\vartheta}_{Ej})\cdot q_j+m'i_m^2\bar{\vartheta}_{Ej}\ddot{q}_j+m'l_{ES}(\bar{y}_{Ej}\sin\vartheta_u-\bar{z}_{Ej}\cos\vartheta_u)\ddot{q}_j\right] = M_e'(x,t,\dot{y}_E,\dot{z}_E,\vartheta_E,\dot{\vartheta}_E)$$

Now we multiply Eq. (4.24) with q_j and sum from $j = 1$ to n:

$$\sum_{j=1}^{n}\left[f(d,\bar{y}_{Ej},\bar{z}_{Ej},\bar{\vartheta}_{Ej})\cdot q_j-\nu_j^2m'\bar{y}_{Ej}q_j-\nu_j^2m'l_{ES}\bar{\vartheta}_{Ej}q_j\sin\vartheta_u\right] = 0$$

$$\sum_{j=1}^{n}\left[g(d,\bar{y}_{Ej},\bar{z}_{Ej},\bar{\vartheta}_{Ej})\cdot q_j-\nu_j^2m'\bar{z}_{Ej}q_j+\nu_j^2m'l_{ES}\bar{\vartheta}_{Ej}q_j\cos\vartheta_u\right] = 0 \qquad (5.34)$$

$$\sum_{j=1}^{n}\left[h(d,\bar{y}_{Ej},\bar{z}_{Ej},\bar{\vartheta}_{Ej})\cdot q_j-\nu_j^2m'i_m^2\bar{\vartheta}_{Ej}q_j-\nu_j^2m'l_{ES}(\bar{y}_{Ej}\sin\vartheta_u-\bar{z}_{Ej}\cos\vartheta_u)q_j\right]=0$$

If Eq. (5.34) is subtracted from Eq. (5.33), the remainder is

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$$\sum_{j=1}^{n}\left[m'(\bar{y}_{Ej}+l_{ES}\bar{\vartheta}_{Ej}\sin\vartheta_u)(\ddot{q}_j+\nu_j^2q_j)\right] = Y_e'(x,t,\dot{y}_E,\dot{z}_E,\vartheta_E)$$

$$\sum_{j=1}^{n}\left[m'(\bar{z}_{Ej}-l_{ES}\bar{\vartheta}_{Ej}\cos\vartheta_u)(\ddot{q}_j+\nu_j^2q_j)\right] = Z_e'(x,t,\dot{y}_E,\dot{z}_E,\vartheta_E,\dot{\vartheta}_E) \qquad (5.35)$$

$$\sum_{j=1}^{n}\left[m'(i_m^2\bar{\vartheta}_{Ej}+l_{ES}\bar{y}_{Ej}\sin\vartheta_u-l_{ES}\bar{z}_{Ej}\cos\vartheta_u)(\ddot{q}_j+\nu_j^2q_j)\right]=M_e'(x,t,\dot{y}_E,\dot{z}_E,\vartheta_E,\dot{\vartheta}_E)$$

We multiply through these equations by $\bar{y}_{Ei}$, $\bar{z}_{Ei}$, and $\bar{\theta}_{Ei}$ respectively, integrate from 0 to R_A and add the equations.

$$\sum_{j=1}^{n}\left(\ddot{q}_j+\nu_j^2 q_j\right)\cdot\int_0^{R_A} m'\Big[\bar{y}_{Ei}\bar{y}_{Ej}+\bar{z}_{Ei}\bar{z}_{Ej}+i_m^2\bar{\vartheta}_{Ei}\bar{\vartheta}_{Ej} +l_{ES}\left(\bar{y}_{Ei}\bar{\vartheta}_{Ej}+\bar{y}_{Ej}\bar{\vartheta}_{Ei}\right)\sin\vartheta_u-l_{ES}\left(\bar{z}_{Ei}\bar{\vartheta}_{Ej}+\bar{z}_{Ej}\bar{\vartheta}_{Ei}\right)\cos\vartheta_u\Big]dx$$
$$=\int_0^{R_A}\Big[Y_e'(x,t,\dot{y}_E,\dot{z}_E,\vartheta_E)\cdot\bar{y}_{Ei}+Z_e'(x,t,\dot{y}_E,\dot{z}_E,\vartheta_E,\dot{\vartheta}_E)\cdot\bar{z}_{Ei}+M_e'(x,t,\dot{y}_E,\dot{z}_E,\vartheta_E,\dot{\vartheta}_E)\cdot\bar{\vartheta}_{Ei}\Big]dx \qquad (5.36)$$

Because of the orthogonality condition (4.29), all terms in the sum on the left side disappear except for the one with j = i. Therefore,

$$\ddot{q}_i+\nu_i^2 q_i = \frac{\displaystyle\int_0^{R_A}\Big[Y_e'(x,t,\dot{y}_E,\dot{z}_E,\vartheta_E)\cdot\bar{y}_{Ei}+Z_e'(x,t,\dot{y}_E,\dot{z}_E,\vartheta_E,\dot{\vartheta}_E)\cdot\bar{z}_{Ei}+M_e'(x,t,\dot{y}_E,\dot{z}_E,\vartheta_E,\dot{\vartheta}_E)\cdot\bar{\vartheta}_{Ei}\Big]dx}{\displaystyle\int_0^{R_A} m'\Big[\bar{y}_{Ei}^2+\bar{z}_{Ei}^2+i_m^2\bar{\vartheta}_{Ei}^2+2l_{ES}\bar{y}_{Ei}\bar{\vartheta}_{Ei}\sin\vartheta_u-2l_{ES}\bar{z}_{Ei}\bar{\vartheta}_{Ei}\cos\vartheta_u\Big]dx} \qquad (5.37)$$

6. Inclusion of Inertia Due to Extension of the Blade in the Transverse Direction and β_{B1}-Terms, Particularly Coriolis Force /37

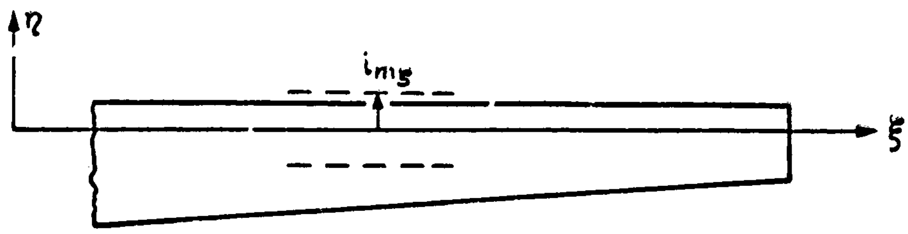

Fig. 6.1. Undeformed rotor blade with representative mass displacement $i_{m\zeta}$.

In the work of Houbolt and Brooks [1], the torque due to angular acceleration and resulting from the extension of the blades in the transverse direction were temporarily ignored. These torques are particularly important in natural vibrations. In this case, we are particularly interested in the blade extension in the η-direction, since it greatly exceeds that the in ζ-direction. We go back in Houbolt and Brooks just to point at which the terms concerned were neglected. The equations for the moments $\bar{m}_z$ and $\bar{m}_y$ now acquire the following additional terms Z when a less drastic simplification is carried out.

$$
\begin{aligned}
Z(\bar{m}_z) &= -m'\cdot(i_{m\xi}^2 \cos^2\vartheta_u + i_{m\eta}^2 \sin^2\vartheta_u)\cdot \ddot{y}_E' \\
&\quad -m'\cdot(i_{m\xi}^2 - i_{m\eta}^2)\sin\vartheta_u \cos\vartheta_u \cdot \ddot{z}_E' \\
Z(\bar{m}_y) &= -m'(i_{m\xi}^2 \sin^2\vartheta_u + i_{m\eta}^2 \cos^2\vartheta_u)\cdot \ddot{z}_E' \\
&\quad -m'(i_{m\xi}^2 - i_{m\eta}^2)\sin\vartheta_u \cos\vartheta_u \cdot \ddot{y}_E'
\end{aligned}
\tag{6.1}
$$

Some simplifications have been retained, but they are justifiable even under the enhanced accuracy requirements. For example, $\omega_{R_o}{}^2 z_E'$ and $2\omega_{R_o}\dot{\theta}_E$ are considered small in comparison with $\ddot{z}_E'$. This is legitimate for the higher natural frequencies ν with which we are specially concerned in this case, since the terms are roughly in the proportion $\omega_{R_o}{}^2$ to $2\omega_{R_o}\nu$ to ν^2. To clarify Eq. (6.1), we set $\theta_u = i_{m\eta} = 0$, and obtain /38

$$
\begin{aligned}
Z(\bar{m}_z) &= -m' i_{m\xi}^2 \ddot{y}_E' \\
Z(\bar{m}_y) &= 0
\end{aligned}
\tag{6.2}
$$

We will not use this equation further, first because θ_u (and perhaps $i_{m\eta}$ as well) may be even somewhat larger, and second because the coefficients of Eq. (6.1) occur more often so that the extra work in comparison with Eq. (6.2) is not very great. At any rate, in Eq. (6.2), we have an expression which can be derived very simply from Fig. 6.1, if the x and y axes are made to coincide with the ξ and η axes respectively.

Since, according to Houbolt and Brooks, $\bar{m}_z'$ and $\bar{m}_y'$ contribute additively to the left sides of the first and second equations respectively in Eq. (3.4), the correction terms $Z'(\bar{m}_z)$ and $Z'(\bar{m}_y)$ from Eq. (6.1) must be added on the left to the first and second equations in (3.4).

So far, we have not mentioned the flapping angle β_{B1}. The deflection due to β_{B1} (with a flapping hinge) should be contained in the z deflections. In order to get some idea of the Coriolis forces in the y-direction, the coordinate system is rotated through an angle of $\beta_{B1}(t)$. This adds a few more terms. We will not bother to derive them at this point. It would require an expansion of the derivation given by Houbolt and Brooks [1], but would not involve any fundamental difficulties. Furthermore, each of the additional terms is easy to comprehend. Now, with the abbreviations introduced in Eqs. (4.23) and (5.31), with the additional terms from Eq. (6.1), and with the additional terms due to the coordinate system being rotated through β_{B1}, Eq. (3.4) now reads: /39

$$
\begin{aligned}
&f(d, y_E, z_E, \vartheta_E) + m'\ddot{y}_E + m' l_{ES}\ddot{\vartheta}_E \sin\vartheta_u - \left[m'(i_{m\xi}^2 \cos^2\vartheta_u + i_{m\eta}^2 \sin^2\vartheta_u)\ddot{y}_E'\right. \\
&\left.+ m'(i_{m\xi}^2 - i_{m\eta}^2)\sin\vartheta_u \cos\vartheta_u \ddot{z}_E'\right]' = Y_e'(x, t, \dot{y}_E, \dot{z}_E, \vartheta_E) \\
&+ 2\omega_{Ro} m'(\dot{z}_E \beta_{Bl} + x\beta_{Bl}\dot{\beta}_{Bl} + z_E\dot{\beta}_{Bl} - l_{ES}\dot{\beta}_{Bl}\sin\vartheta_u)
\end{aligned}
$$

$$
\begin{aligned}
&g(d, y_E, z_E, \vartheta_E) + m'\ddot{z}_E - m' l_{ES}\ddot{\vartheta}_E \cos\vartheta_u - \left[m'(i_{m\xi}^2 \sin^2\vartheta_u + i_{m\eta}^2 \cos^2\vartheta_u)\ddot{z}_E'\right. \\
&\left.+ m'(i_{m\xi}^2 - i_{m\eta}^2)\sin\vartheta_u \cos\vartheta_u \ddot{y}_E'\right]' = Z_e'(x, t, \dot{y}_E, \dot{z}_E, \vartheta_E, \dot{\vartheta}_E) \qquad (6.3) \\
&- m'\left[x\ddot{\beta}_{Bl} + 2\omega_{Ro}\dot{y}_E\beta_{Bl} + \omega_{Ro}^2(x+a)\beta_{Bl}\right]
\end{aligned}
$$

$$
\begin{aligned}
&h(d, y_E, z_E, \vartheta_E) + m' i_m^2 \ddot{\vartheta}_E + m' l_{ES}(\ddot{y}_E \sin\vartheta_u - \ddot{z}_E \cos\vartheta_u) \\
&= M_e'(x, t, \dot{y}_E, \dot{z}_E, \vartheta_E, \dot{\vartheta}_E) + m'\left[x\ddot{\beta}_{Bl} + 2\omega_{Ro}\dot{y}_E\beta_{Bl} + \omega_{Ro}^2(x+a)\beta_{Bl}\right] l_{ES}\cos\vartheta_u \\
&- m'\left[x\ddot{\beta}_{Bl} + \omega_{Ro}^2(x+a)\beta_{Bl}\right] l_{ES}\vartheta_E \sin\vartheta_u + 2\omega_{Ro} m'(\dot{z}_E\beta_{Bl} + x\beta_{Bl}\dot{\beta}_{Bl} + z_E\dot{\beta}_{Bl}) l_{ES}\sin\vartheta_u
\end{aligned}
$$

It should also be observed at this point that the centrifugal tension F_{xF} again depends on β_{B1} (the factors $\cos\beta_{B1}$). To simplify things, we wish to omit the underlined terms in Eq. (6.3). They are third-order terms, if β_{B1}, z_E, θ_E, and l_{ES} and all their derivatives are considered small (or fourth-order terms, when θ_u is small as well). Hence, the β_{B1}-dependence of P_{XF} can be neglected in the abbreviations f, g, and h as well. If we did not get rid of these terms, we would have to deal with natural frequencies which depended on β_{B1}, and therefore on time (the influence on the natural modes is even smaller), or we would have to insert corresponding correction terms on the right-hand side, e.g.

$$(P_{xF}\, y_E')' \cdot (1 - \cos\beta_{B1}).$$

The magnitude of this simplification can be diminished by computing the natural vibrations with the mean value of β_{B1}. However, in ordinary cases with $a_o \sim 5°$, it is hardly worth it.

The dependence of P_{XF} on β_{B1} can easily be taken into account /40 in the abbreviations Y'_e, Z'_e, and M'_e, although there is not much benefit in this either. Eq. (6.3) now becomes

$$
\begin{aligned}
&f(d, y_E, z_E, \vartheta_E) + m'\ddot{y}_E + m' l_{ES}\ddot{\vartheta}_E \sin\vartheta_u - \left[m'(i_{m\xi}^2\cos^2\vartheta_u + i_{m\eta}^2\sin^2\vartheta_u)\ddot{y}_E' \right.\\
&\left. + m'(i_{m\xi}^2 - i_{m\eta}^2)\sin\vartheta_u\cos\vartheta_u\ddot{z}_E'\right]' = Y_{ee}'(x, t, \dot{y}_E, \dot{z}_E, \vartheta_E, \beta_{B1}, \dot{\beta}_{B1})\\
&g(d, y_E, z_E, \vartheta_E) + m'\ddot{z}_E - m' l_{ES}\ddot{\vartheta}_E\cos\vartheta_u - \left[m'(i_{m\xi}^2\sin^2\vartheta_u + i_{m\eta}^2\cos^2\vartheta_u)\ddot{z}_E' \right.\\
&\left. + m'(i_{m\xi}^2 - i_{m\eta}^2)\sin\vartheta_u\cos\vartheta_u\ddot{y}_E'\right]' = Z_{ee}'(x, t, \dot{y}_E, \dot{z}_E, \vartheta_E, \dot{\vartheta}_E, \beta_{B1}, \dot{\beta}_{B1})\\
&h(d, y_E, z_E, \vartheta_E) + m' i_m^2\ddot{\vartheta}_E + m' l_{ES}(\ddot{y}_E\sin\vartheta_u - \ddot{z}_E\cos\vartheta_u)\\
&= M_{ee}'(x, t, \dot{y}_E, \dot{z}_E, \vartheta_E, \dot{\vartheta}_E, \beta_{B1}, \ddot{\beta}_{B1})
\end{aligned}
\tag{6.4}
$$

with

$$
\begin{aligned}
Y_{ee}'(x, t, \dot{y}_E, \dot{z}_E, \vartheta_E, \beta_{B1}, \dot{\beta}_{B1}) &= Y_e'(x, t, \dot{y}_E, \dot{z}_E, \vartheta_E)\\
&+ 2\omega_{Ro} m'(\dot{z}_E\beta_{B1} + x\beta_{B1}\dot{\beta}_{B1} + z_E\dot{\beta}_{B1} - l_{ES}\dot{\beta}_{B1}\sin\vartheta_u)
\end{aligned}
$$

$$Z_{ee}'(x,t,\dot{y}_E,\dot{z}_E,\vartheta_E,\dot{\vartheta}_E,\beta_{Bl},\ddot{\beta}_{Bl}) = Z_e'(x,t,\dot{y}_E,\dot{z}_E,\vartheta_E,\dot{\vartheta}_E)$$

$$-m'\left[x\ddot{\beta}_{Bl}+2\omega_{Ro}\dot{y}_E\,\beta_{Bl}+\omega_{Ro}^2(x+a)\beta_{Bl}\right]$$

$$M_{ee}'(x,t,\dot{y}_E,\dot{z}_E,\vartheta_E,\dot{\vartheta}_E,\beta_{Bl},\ddot{\beta}_{Bl}) = M_e'(x,t,\dot{y}_E,\dot{z}_E,\vartheta_E,\dot{\vartheta}_E) \tag{6.5}$$

$$+m'\left[x\ddot{\beta}_{Bl}+2\omega_{Ro}\dot{y}_E\,\beta_{Bl}+\omega_{Ro}^2(x+a)\beta_{Bl}\right]l_{ES}\cos\vartheta_u$$

We also wish to formulate the orthogonality condition for Eq. (6.4). We will follow the discussion in Chapter 4, where the orthogonality condition for Eq. (3.4) was formulated. Again we use the natural modes defined in Eq. (3.8), although these modes will appear somewhat different numerically, since not Eq. (3.4) but Eq. (6.4) with a vanishing right side was used to calculate them. Eq. (4.23) now becomes

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$$\begin{aligned}
&f(d,y_{Ej}^*,z_{Ej}^*,\vartheta_{Ej}^*)+m'\ddot{y}_{Ej}^*+m'l_{ES}\ddot{\vartheta}_{Ej}^*\sin\vartheta_u-(m'i_{mc}^2\ddot{y}_{Ej}^{*\prime}+m'i_{mo}^2\ddot{z}_{Ej}^{*\prime})'=0\\
&g(d,y_{Ej}^*,z_{Ej}^*,\vartheta_{Ej}^*)+m'\ddot{z}_{Ej}^*-m'l_{ES}\ddot{\vartheta}_{Ej}^*\cos\vartheta_u-(m'i_{ms}^2\ddot{z}_{Ej}^{*\prime}+m'i_{mo}^2\ddot{y}_{Ej}^{*\prime})'=0\\
&h(d,y_{Ej}^*,z_{Ej}^*,\vartheta_{Ej}^*)+m'i_m^2\ddot{\vartheta}_{Ej}^*+m'l_{ES}(\ddot{y}_{Ej}^*\sin\vartheta_u-\ddot{z}_{Ej}^*\cos\vartheta_u)=0
\end{aligned} \tag{6.6}$$

With the abbreviations

$$\begin{aligned}
i_{mc}^2 &= i_{m\xi}^2\cos^2\vartheta_u+i_{m\eta}^2\sin^2\vartheta_u\\
i_{mo}^2 &= (i_{m\xi}^2-i_{m\eta}^2)\sin\vartheta_u\cos\vartheta_u\\
i_{ms}^2 &= i_{m\xi}^2\sin^2\vartheta_u+i_{m\eta}^2\cos^2\vartheta_u
\end{aligned} \tag{6.7}$$

Analogously, Eq. (4.24) becomes

$$
\begin{aligned}
& f(d, \bar{y}_{Ej}, \bar{z}_{Ej}, \bar{\vartheta}_{Ej}) - \nu_j^2\left[m'\bar{y}_{Ej} + m' l_{ES}\bar{\vartheta}_{Ej}\sin\vartheta_u - (m' i_{mc}^2 \bar{y}_{Ej}' + m' i_{mo}^2 \bar{z}_{Ej}')'\right] = 0 \\
& g(d, \bar{y}_{Ej}, \bar{z}_{Ej}, \bar{\vartheta}_{Ej}) - \nu_j^2\left[m'\bar{z}_{Ej} - m' l_{ES}\bar{\vartheta}_{Ej}\cos\vartheta_u - (m' i_{ms}^2 \bar{z}_{Ej}' + m' i_{mo}^2 \bar{y}_{Ej}')'\right] = 0 \qquad (6.8) \\
& h(d, \bar{y}_{Ej}, \bar{z}_{Ej}, \bar{\vartheta}_{Ej}) - \nu_j^2\left[m' i_m^2 \bar{\vartheta}_{Ej} + m' l_{ES}(\bar{y}_{Ej}\sin\vartheta_u - \bar{z}_{Ej}\cos\vartheta_u)\right] = 0
\end{aligned}
$$

For j = p and j = q, Eq. (6.8) becomes

$$
\begin{aligned}
& f(d, \bar{y}_{Ep}, \bar{z}_{Ep}, \bar{\vartheta}_{Ep}) - \nu_p^2\left[m'\bar{y}_{Ep} + m' l_{ES}\bar{\vartheta}_{Ep}\sin\vartheta_u - (m' i_{mc}^2 \bar{y}_{Ep}' + m' i_{mo}^2 \bar{z}_{Ep}')'\right] = 0 \\
& g(d, \bar{y}_{Ep}, \bar{z}_{Ep}, \bar{\vartheta}_{Ep}) - \nu_p^2\left[m'\bar{z}_{Ep} - m' l_{ES}\bar{\vartheta}_{Ep}\cos\vartheta_u - (m' i_{ms}^2 \bar{z}_{Ep}' + m' i_{mo}^2 \bar{y}_{Ep}')'\right] = 0 \\
& h(d, \bar{y}_{Ep}, \bar{z}_{Ep}, \bar{\vartheta}_{Ep}) - \nu_p^2\left[m' i_m^2 \bar{\vartheta}_{Ep} + m' l_{ES}(\bar{y}_{Ep}\sin\vartheta_u - \bar{z}_{Ep}\cos\vartheta_u)\right] = 0 \qquad (6.9) \\
& f(d, \bar{y}_{Eq}, \bar{z}_{Eq}, \bar{\vartheta}_{Eq}) - \nu_q^2\left[m'\bar{y}_{Eq} + m' l_{ES}\bar{\vartheta}_{Eq}\sin\vartheta_u - (m' i_{mc}^2 \bar{y}_{Eq}' + m' i_{mo}^2 \bar{z}_{Eq}')'\right] = 0 \\
& g(d, \bar{y}_{Eq}, \bar{z}_{Eq}, \bar{\vartheta}_{Eq}) - \nu_q^2\left[m'\bar{z}_{Eq} - m' l_{ES}\bar{\vartheta}_{Eq}\cos\vartheta_u - (m' i_{ms} \bar{z}_{Eq}' + m' i_{mo} \bar{y}_{Eq}')'\right] = 0 \\
& h(d, \bar{y}_{Eq}, \bar{z}_{Eq}, \bar{\vartheta}_{Eq}) - \nu_q^2\left[m' i_m^2 \bar{\vartheta}_{Eq} + m' l_{ES}(\bar{y}_{Eq}\sin\vartheta_u - \bar{z}_{Eq}\cos\vartheta_u)\right] = 0
\end{aligned}
$$

We multiply through by $\bar{y}_{Eq}$, $\bar{z}_{Eq}$, $\bar{\theta}_{Eq}$, $-\bar{y}_{Ep}$, $-\bar{z}_{Ep}$, and $-\bar{\theta}_{Ep}$ then add all six equations, and integrate over the length of the blade: /42

$$
\begin{aligned}
& \int_0^{R_A} \Big[\bar{y}_{Eq}\cdot f(d, \bar{y}_{Ep}, \bar{z}_{Ep}, \bar{\vartheta}_{Ep}) - \bar{y}_{Ep}\cdot f(d, \bar{y}_{Eq}, \bar{z}_{Eq}, \bar{\vartheta}_{Eq}) \\
& \quad + \bar{z}_{Eq}\cdot g(d, \bar{y}_{Ep}, \bar{z}_{Ep}, \bar{\vartheta}_{Ep}) - \bar{z}_{Ep}\cdot g(d, \bar{y}_{Eq}, \bar{z}_{Eq}, \bar{\vartheta}_{Eq}) \\
& \quad + \bar{\vartheta}_{Eq}\cdot h(d, \bar{y}_{Ep}, \bar{z}_{Ep}, \bar{\vartheta}_{Ep}) - \bar{\vartheta}_{Ep}\cdot h(d, \bar{y}_{Eq}, \bar{z}_{Eq}, \bar{\vartheta}_{Eq})\Big] dx \\
& = (\nu_p^2 - \nu_q^2) \int_0^{R_A} m' \Big[\bar{y}_{Ep}\bar{y}_{Eq} + \bar{z}_{Ep}\bar{z}_{Eq} + i_m^2 \bar{\vartheta}_{Ep}\bar{\vartheta}_{Eq} \qquad (6.10) \\
& \qquad + l_{ES}(\bar{y}_{Ep}\bar{\vartheta}_{Eq} + \bar{y}_{Eq}\bar{\vartheta}_{Ep})\sin\vartheta_u - l_{ES}(\bar{z}_{Ep}\bar{\vartheta}_{Eq} + \bar{z}_{Eq}\bar{\vartheta}_{Ep})\cos\vartheta_u\Big] dx \\
& - \int_0^{R_A} \Big\{\nu_p^2\Big[(m' i_{mc}^2 \bar{y}_{Ep}' + m' i_{mo}^2 \bar{z}_{Ep}')'\bar{y}_{Eq} + (m' i_{ms}^2 \bar{z}_{Ep}' + m' i_{mo}^2 \bar{y}_{Ep}')'\bar{z}_{Eq}\Big] \\
& \qquad - \nu_q^2\Big[(m' i_{mc}^2 \bar{y}_{Eq}' + m' i_{mo}^2 \bar{z}_{Eq}')'\bar{y}_{Ep} + (m' i_{ms}^2 \bar{z}_{Eq}' + m' i_{mo}^2 \bar{y}_{Eq}')'\bar{z}_{Ep}\Big]\Big\} dx
\end{aligned}
$$

The left side of Eq. (6.10) is equal to zero, since it is identical with the left side of Eq. (4.26), and the latter was proven to be zero by using the boundary conditions. By integration by parts, we rewrite the last two lines of Eq. (6.10) as follows:

$$\begin{aligned}
&\int_0^{R_A}\Big\{\nu_p^2\Big[\big(m' i_{mc}^2 \bar{y}_{Ep}' + m' i_{mo}^2 \bar{z}_{Ep}'\big)'\bar{y}_{Eq} + \big(m' i_{ms}^2 \bar{z}_{Ep}' + m' i_{mo}^2 \bar{y}_{Ep}'\big)'\bar{z}_{Eq}\Big] \\
&\quad - \nu_q^2\Big[\big(m' i_{mc}^2 \bar{y}_{Eq}' + m' i_{mo}^2 \bar{z}_{Eq}'\big)'\bar{y}_{Ep} + \big(m' i_{ms}^2 \bar{z}_{Eq}' + m' i_{mo}^2 \bar{y}_{Eq}'\big)'\bar{z}_{Ep}\Big]\Big\}dx \\
&= \Big\{\nu_p^2\Big[\big(m' i_{mc}^2 \bar{y}_{Ep}' + m' i_{mo}^2 \bar{z}_{Ep}'\big)\bar{y}_{Eq} + \big(m' i_{ms}^2 \bar{z}_{Ep}' + m' i_{mo} \bar{y}_{Ep}'\big)\bar{z}_{Eq}\Big] \\
&\quad - \nu_q^2\Big[\big(m' i_{mc}^2 \bar{y}_{Ep}' + m' i_{mo}^2 \bar{z}_{Eq}'\big)\bar{y}_{Ep} + \big(m' i_{ms}^2 \bar{z}_{Eq}' + m' i_{mo}^2 \bar{y}_{Eq}'\big)\bar{z}_{Ep}\Big]\Big\}\Big|_0^{R_A} \\
&\quad - \int_0^{R_A}(\nu_p^2 - \nu_q^2)\, m'\Big[i_{mc}^2 \bar{y}_{Ep}' \bar{y}_{Eq}' + i_{mo}^2\big(\bar{y}_{Ep}'\bar{z}_{Eq}' + \bar{z}_{Ep}'\bar{y}_{Eq}'\big) + i_{ms}^2 \bar{z}_{Ep}'\bar{z}_{Eq}'\Big]dx
\end{aligned} \qquad (6.11)$$

The expression inside the second pair of wavy brackets in Eq. (6.11) is equal to zero, since y_E and z_E vanish at $x = 0$, and m' vanishes at $x = R_A$. Eq. (6.10) now yields the orthogonality relation /43

$$\boxed{\begin{aligned}
&\int_0^{R_A} m'\Big[\bar{y}_{Ep}\bar{y}_{Eq} + \bar{z}_{Ep}\bar{z}_{Eq} + i_m^2 \bar{\vartheta}_{Ep}\bar{\vartheta}_{Eq} \\
&\quad + l_{ES}\big(\bar{y}_{Ep}\bar{\vartheta}_{Eq} + \bar{y}_{Eq}\bar{\vartheta}_{Ep}\big)\sin\vartheta_u - l_{ES}\big(\bar{z}_{Ep}\bar{\vartheta}_{Eq} + \bar{z}_{Eq}\bar{\vartheta}_{Ep}\big)\cos\vartheta_u \\
&\quad + i_{mc}^2 \bar{y}_{Ep}'\bar{y}_{Eq}' + i_{mo}^2\big(\bar{y}_{Ep}'\bar{z}_{Eq}' + \bar{z}_{Ep}'\bar{y}_{Eq}'\big) + i_{ms}^2 \bar{z}_{Ep}'\bar{z}_{Eq}'\Big]dx = 0 \\
&\qquad\qquad \text{for } \nu_p \neq \nu_q
\end{aligned}} \qquad (6.12)$$

By analogy with Chapter 5.4, the solution of Eq. (6.4) is organized as follows. With the trial-solution sum in Eq. (5.32), Eq. (6.4) becomes (since f, g, and h are linear functions of the unknown)

$$\sum_{j=1}^{n}\Big[f(d,\bar{y}_{Ej},\bar{z}_{Ej},\bar{\vartheta}_{Ej})q_j+m'\bar{y}_{Ej}\ddot{q}_j+m'l_{ES}\bar{\vartheta}_{Ej}\ddot{q}_j\sin\vartheta_u-(m'i_{mc}^2\bar{y}_{Ej}'+m'i_{mo}^2\bar{z}_{Ej}')'\ddot{q}_j\Big]=Y_{ee}'(x,t,\dot{y}_E,\dot{z}_E,\vartheta_E,\beta_{Bl},\dot{\beta}_{Bl})$$

$$\sum_{j=1}^{n}\Big[g(d,\bar{y}_{Ej},\bar{z}_{Ej},\bar{\vartheta}_{Ej})q_j+m'\bar{z}_{Ej}\ddot{q}_j-m'l_{ES}\bar{\vartheta}_{Ej}\ddot{q}_j\cos\vartheta_u-(m'i_{ms}^2\bar{z}_{Ej}'+m'i_{mo}^2\bar{y}_{Ej}')'\ddot{q}_j\Big]=Z_{ee}'(x,t,\dot{y}_E,\dot{z}_E,\vartheta_E,\dot{\vartheta}_E,\beta_{Bl},\ddot{\beta}_{Bl}) \qquad (6.13)$$

$$\sum_{j=1}^{n}\Big[h(d,\bar{y}_{Ej},\bar{z}_{Ej},\bar{\vartheta}_{Ej})q_j+m'i_m^2\bar{\vartheta}_{Ej}\ddot{q}_j+m'l_{ES}(\bar{y}_{Ej}\sin\vartheta_u-\bar{z}_{Ej}\cos\vartheta_u)\ddot{q}_j\Big]=M_{ee}'(x,t,\dot{y}_E,\dot{z}_E,\vartheta_E,\dot{\vartheta}_E,\beta_{Bl},\ddot{\beta}_{Bl})$$

with the abbreviations i_{mc}, i_{mo} and i_{ms} as in Eq. (6.7). We multiply Eq. (6.8) by q_j and sum from $j = 1$ to n. /44

$$\sum_{j=1}^{n}\Big[f(d,\bar{y}_{Ej},\bar{z}_{Ej},\bar{\vartheta}_{Ej})q_j-\nu_j^2m'\bar{y}_{Ej}q_j-\nu_j^2m'l_{ES}\bar{\vartheta}_{Ej}q_j\sin\vartheta_u+\nu_j^2(m'i_{mc}^2\bar{y}_{Ej}'+m'i_{mo}^2\bar{z}_{Ej}')'q_j\Big]=0$$

$$\sum_{j=1}^{n}\Big[g(d,\bar{y}_{Ej},\bar{z}_{Ej},\bar{\vartheta}_{Ej})q_j-\nu_j^2m'\bar{z}_{Ej}q_j+\nu_j^2m'l_{ES}\bar{\vartheta}_{Ej}q_j\cos\vartheta_u+\nu_j^2(m'i_{ms}^2\bar{z}_{Ej}'+m'i_{mo}^2\bar{y}_{Ej}')'q_j\Big]=0 \qquad (6.14)$$

$$\sum_{j=1}^{n}\Big[h(d,\bar{y}_{Ej},\bar{z}_{Ej},\bar{\vartheta}_{Ej})\cdot q_j-\nu_j^2m'i_m^2\bar{\vartheta}_{Ej}q_j-\nu_j^2m'l_{ES}(\bar{y}_{Ej}\sin\vartheta_u-\bar{z}_{Ej}\cos\vartheta_u)q_j\Big]=0$$

The difference between Eqs. (6.13) and (6.14) is

$$\sum_{j=1}^{n}\Big\{\Big[m'\bar{y}_{Ej}+m'l_{ES}\bar{\vartheta}_{Ej}\sin\vartheta_u-(m'i_{mc}^2\bar{y}_{Ej}'+m'i_{mo}^2\bar{z}_{Ej}')'\Big]\cdot(\ddot{q}_j+\nu_j^2q_j)\Big\}=Y_{ee}'(x,t,\dot{y}_E,\dot{z}_E,\vartheta_E,\beta_{Bl},\dot{\beta}_{Bl})$$

$$\sum_{j=1}^{n}\Big\{\Big[m'\bar{z}_{Ej}-m'l_{ES}\bar{\vartheta}_{Ej}\cos\vartheta_u-(m'i_{ms}^2\bar{z}_{Ej}'+m'i_{mo}^2\bar{y}_{Ej}')'\Big]\cdot(\ddot{q}_j+\nu_j^2q_j)\Big\}=Z_{ee}'(x,t,\dot{y}_E,\dot{z}_E,\vartheta_E,\dot{\vartheta}_E,\beta_{Bl},\ddot{\beta}_{Bl}) \qquad (6.15)$$

$$\sum_{j=1}^{n}\Big\{m'\Big[i_m^2\bar{\vartheta}_{Ej}+l_{ES}(\bar{y}_{Ej}\sin\vartheta_u-\bar{z}_{Ej}\cos\vartheta_u)\Big]\cdot(\ddot{q}_j+\nu_j^2q_j)\Big\}=M_{ee}'(x,t,\dot{y}_E,\dot{z}_E,\vartheta_E,\dot{\vartheta}_E,\beta_{Bl},\ddot{\beta}_{Bl})$$

These equations are multiplied through by $\bar{y}_{E1}$, $\bar{z}_{E1}$ and $\bar{\theta}_{E1}$, integrated from 0 to R_A, and added.

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$$\sum_{j=1}^{n}(\ddot{q}_j+\nu_j^2 q_j)\cdot\int_0^{R_A}\Big\{m'\big[\bar{y}_{Ei}\,\bar{y}_{Ej}+\bar{z}_{Ei}\,\bar{z}_{Ej}+i_m^2\,\bar{\vartheta}_{Ei}\,\bar{\vartheta}_{Ej}$$
$$+l_{ES}(\bar{y}_{Ei}\,\bar{\vartheta}_{Ej}+\bar{y}_{Ej}\,\bar{\vartheta}_{Ei})\sin\vartheta_u-l_{ES}(\bar{z}_{Ei}\,\bar{\vartheta}_{Ej}+\bar{z}_{Ej}\,\bar{\vartheta}_{Ei})\cos\vartheta_u\big] \tag{6.16}$$
$$-(m'i_{mc}^2\,\bar{y}_{Ej}'+m'i_{mo}^2\,\bar{z}_{Ej}')'\bar{y}_{Ei}-(m'i_{ms}^2\,\bar{z}_{Ej}'+m'i_{mo}^2\,\bar{y}_{Ej}')'\bar{z}_{Ei}\Big\}dx$$
$$=\int_0^{R_A}\big[Y_{ee}'\cdot\bar{y}_{Ei}+Z_{ee}'\cdot\bar{z}_{Ei}+M_{ee}'\cdot\bar{\vartheta}_{Ei}\big]dx$$

The third line in this equations must be reformulated by integration by parts. Like Eq. (6.11), we have

$$\int_0^{R_A}\Big\{(m'i_{mc}^2\,\bar{y}_{Ej}'+m'i_{mo}^2\,\bar{z}_{Ej}')'\bar{y}_{Ei}+(m'i_{ms}^2\,\bar{z}_{Ej}'+m'i_{mo}^2\,\bar{y}_{Ej}')'\bar{z}_{Ei}\Big\}dx \tag{6.17}$$
$$=-\int_0^{R_A}m'\big[i_{mc}^2\,\bar{y}_{Ei}'\,\bar{y}_{Ej}'+i_{mo}^2(\bar{y}_{Ei}'\,\bar{z}_{Ej}'+\bar{z}_{Ei}'\,\bar{y}_{Ej}')+i_{ms}^2\,\bar{z}_{Ei}'\,\bar{z}_{Ej}'\big]dx$$

With the aid of Eq. (6.17), Eq. (6.16) becomes

$$\sum_{j=1}^{n}(\ddot{q}_j+\nu_j^2 q_j)\cdot\int_0^{R_A}m'\big[\bar{y}_{Ei}\,\bar{y}_{Ej}+\bar{z}_{Ei}\,\bar{z}_{Ej}+i_m^2\,\bar{\vartheta}_{Ei}\,\bar{\vartheta}_{Ej}$$
$$+l_{ES}(\bar{y}_{Ei}\,\bar{\vartheta}_{Ej}+\bar{y}_{Ej}\,\bar{\vartheta}_{Ei})\sin\vartheta_u-l_{ES}(\bar{z}_{Ei}\,\bar{\vartheta}_{Ej}+\bar{z}_{Ej}\,\bar{\vartheta}_{Ei})\cos\vartheta_u$$
$$+i_{mc}^2\,\bar{y}_{Ei}'\,\bar{y}_{Ej}'+i_{mo}^2(\bar{y}_{Ei}'\,\bar{z}_{Ej}'+\bar{z}_{Ei}'\,\bar{y}_{Ej}')+i_{ms}^2\,\bar{z}_{Ei}'\,\bar{z}_{Ej}'\big]dx \tag{6.18}$$
$$=\int_0^{R_A}\big[Y_{ee}'\cdot\bar{y}_{Ei}+Z_{ee}'\cdot\bar{z}_{Ei}+M_{ee}'\cdot\bar{\vartheta}_{Ei}\big]dx$$

Because of the orthogonality condition (6.12), all the terms in the sum on the left side disappear except for the one with $j = i$. Therefore,

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$$\ddot{q}_i + \nu_i^2 q_i = \frac{\int_0^{R_A} \left[Y_{ee}' \cdot \bar{y}_{Ei} + Z_{ee}' \cdot \bar{z}_{Ei} + M_{ee}' \cdot \bar{\vartheta}_{Ei} \right] dx}{\int_0^{R_A} m' \left[\bar{y}_{Ei}^2 + \bar{z}_{Ei}^2 + i_m^2 \bar{\vartheta}_{Ei}^2 + 2 l_{ES} (\bar{y}_{Ei} \bar{\vartheta}_{Ei} \sin\vartheta_U - \bar{z}_{Ei} \bar{\vartheta}_{Ei} \cos\vartheta_U) + i_{mc}^2 \bar{y}_{Ei}'^2 + 2 i_{mo}^2 \bar{y}_{Ei}' \bar{z}_{Ei}' + i_{ms}^2 \bar{z}_{Ei}'^2 \right] dx} \qquad (6.19)$$

As is evident, the expression in the denominator, i.e. the generalized mass, is larger than that in Eq. (5.37) by the last three terms. These terms are particularly important in high-order natural modes, in which y_E' and z_E' are relatively large in comparison with y_E and z_E. The ν_i^2 must be correspondingly smaller than the ν_i^2 in Eq. (5.37), while the behavior of $\bar{y}_E$, $\bar{z}_E$ and $\bar{\theta}_E$ should hardly change at all.

The β_{B1} contained in Y_{ee}', Z_{ee}', and M_{ee}' is an additional degree of freedom apart from q_1 through q_n. In order to determine it, we need an additional condition. It is well known that β_{B1} determines only the position of the coordinate system (see Fig. 2.1). In that diagram, a flapping hinge was assumed. Computing with a flapping angle without a flapping hinge (at a purely imaginary, spring-loaded flapping hinge, by analogy with the method of Payne) gives rise to several other not very simple problems in the natural-mode method. The blade vibrations we are looking for will be superimposed on the coordinate system rotated through the time-dependent angle β_{B1}, a z_E-deflection no longer being parallel to the rotor axis, so that $\dot{z}_E$ would yield a Coriolis-force component. It is now evident that $\beta_{B1}(t)$ should be such that the x-axis of the coordinate system is the middle chord of the bending curve at all times if possible.

Two methods of calculation suggest themselves. The first is to set $\beta_{B1} = q_1/R$. For a hinged blade, the first natural mode does satisfy the equation $(y_E\ z_E\ \theta_E) = (0\ x/R\ 0)$, within a certain approximation. Hence, q_1 is calculated with the coordinate system unrotated, then β_{B1} is calculated, followed by the remaining q_i in the coordinate system rotated through β_{B1}. (With all the q coupled, the "followed by" is not to be taken literally. Instead, the calculation must naturally be done simultaneously.) /47

The second, more accurate method is to carry out the entire calculation first for $\beta_{B1} = 0$, then calculate the time behavior of

β_{B1} from the computed $q_1(t)$, and then carry out the calculation once more. The values for q_1 will still be small, and these values can then, if necessary, be used for correcting the β_{B1} curve and for carrying out a renewed calculation.

7. Determining q_i and $\dot{q}_i$ at the Start from the Initial Conditions /48

7.1. Initial Values for Uncoupled Flapwise Bending

In order to complete the theory of the natural-mode method, we will now deal with the following problem: given the deflections and their time derivatives at time t = 0, what are q_i and $\dot{q}_i$ at time t = 0? These quantities must be known at the beginning of a stepwise calculation (e.g. the Runge-Kutta method). It is true that the initial values of q_i and $\dot{q}_i$ can be estimated or just set equal to zero. However, if we wish to start from a precise initial state as defined by the deflections, the following transformation of the initial conditions is indispensable. By means of an inverse transformation, through which the values of y_E, z_E, and θ_E, as well as $\dot{y}_E$, $\dot{z}_E$, and $\dot{\theta}_E$ can be recovered from q_i and $\dot{q}_i$, the accuracy of the representation of deflection curves by natural modes can be reviewed. Naturally, complicated bending and torsion curves can be approximated well only by a large number of natural modes.

Again, we begin with the simple case of uncoupled flapwise bending. Let the initial deflection in the z-direction z_{Ea} and its time derivative $\dot{z}_{Ea}$ be given:

$$z_{Ea} = z_{Ea}(x), \quad \dot{z}_{Ea} = \dot{z}_{Ea}(x); \quad a = \text{initial state.}$$

z_{Ea} must be built up from the natural modes $\bar{z}_{Ej}$ in accordance with Eq. (3.5), i.e.

$$z_{Ea}(x) = \sum_{j=1}^{\infty} q_{ja} \cdot \bar{z}_{Ej}(x) \tag{7.1}$$

In order to determine the q_{ja}, we multiply on the left and right by $m'\bar{z}_{Ei}(x)$ and integrate over the blade:

$$\int_0^{R_A} m' z_{Ea} \bar{z}_{Ei} \, dx = \sum_{j=1}^{\infty} q_{ja} \int_0^{R_A} m' \bar{z}_{Ei} \bar{z}_{Ej} \, dx \tag{7.2}$$

Because of the orthogonality condition (4.8), the expression on the right vanishes except when $j = i$. Therefore, /49

$$q_{ia} = \frac{\int_0^{R_A} m' z_{Ea} \bar{z}_{Ei} \, dx}{\int_0^{R_A} m' \bar{z}_{Ei}^2 \, dx} \tag{7.3}$$

This equation is very similar to Eq. (5.8). Analogously, the initial deflection rate $\dot{z}_{Ea}(x)$ yields

$$\dot{q}_{ia} = \frac{\int_0^{R_A} m' \dot{z}_{Ea} \bar{z}_{Ei} \, dx}{\int_0^{R_A} m' \bar{z}_{Ei}^2 \, dx} \tag{7.4}$$

7.2. Initial Value for Coupled Flapwise and Edgewise Bending

Let the initial conditions y_{Ea} and z_{Ea} and their first time derivatives be given. We again form a trial solution with the (two-dimensional) natural modes defined in Eq. (3.6).

$$\begin{bmatrix} y_{Ea} \\ z_{Ea} \end{bmatrix} = \sum_{j=1}^{\infty} q_{ja} \cdot \begin{bmatrix} \bar{y}_{Ej} \\ \bar{z}_{Ej} \end{bmatrix} \tag{7.5}$$

We multiply on top by $m'\bar{y}_{Ei}$, and on the bottom by $m'\bar{z}_{Ei}$, integrate over the blade, and add the two equations.

$$\int_0^{R_A} m'(y_{Ea}\bar{y}_{Ei} + z_{Ea}\bar{z}_{Ei}) \, dx = \sum_{j=1}^{\infty} q_{ja} \int_0^{R_A} m'(\bar{y}_{Ei}\bar{y}_{Ej} + \bar{z}_{Ei}\bar{z}_{Ej}) \, dx \tag{7.6}$$

Because of the orthogonality condition (4.15), all terms in the sum vanish except for that with $j = i$. Thus, by analogy with Eq. (5.21), we obtain

$$q_{ia} = \frac{\int_0^{R_A} m'(y_{Ea}\bar{y}_{Ei} + z_{Ea}\bar{z}_{Ei})\,dx}{\int_0^{R_A} m'(\bar{y}_{Ei}^2 + \bar{z}_{Ei}^2)\,dx} \tag{7.7}$$ /50

Accordingly

$$\dot{q}_{ia} = \frac{\int_0^{R_A} m'(\dot{y}_{Ea}\bar{y}_{Ei} + \dot{z}_{Ea}\bar{z}_{Ei})\,dx}{\int_0^{R_A} m'(\bar{y}_{Ei}^2 + \bar{z}_{Ei}^2)\,dx} \tag{7.8}$$

7.3. Initial Values for Coupled Flapwise Bending, Edgewise Bending and Torsion

We will now proceed directly to the most complicated case discussed in Chapter 6. It contains the two simpler cases discussed in Chapters 5.3 and 5.4 as special cases. To represent the prescribed initial deflections $y_{Ea}(x)$, $z_{Ea}(x)$, and $\theta_{Ea}(x)$, we now form the following trial solutions using the (three-dimensional) natural modes defined in Eq. (3.8).

$$\begin{bmatrix} y_{Ea} \\ z_{Ea} \\ \vartheta_{Ea} \end{bmatrix} = \sum_{j=1}^{\infty} q_{ja} \begin{bmatrix} \bar{y}_{Ej} \\ \bar{z}_{Ej} \\ \bar{\vartheta}_{Ej} \end{bmatrix} \tag{7.9}$$

Following the sum symbol on the right side, we must arrange to have an expression which again vanishes for all j except for j = i. The complicated orthogonality condition (6.12) may be of assistance in making them vanish. The problem is to multiply the three above equations in succession by appropriately chosen factors (in some cases, after first differentiating by x) so that when the resulting equations are added and integrated from 0 to R_A, the expression inside the sum will correspond to that in Eq. (6.12). This does not prove to be especially difficult, and lead to the following equation. /51

$$\int_0^{R_A} m'\Big[y_{Ea}\bar{y}_{Ei}+z_{Ea}\bar{z}_{Ei}+i_m^2\vartheta_{Ea}\bar{\vartheta}_{Ei}+l_{Es}(\bar{y}_{Ei}\vartheta_{Ea}+y_{Ea}\bar{\vartheta}_{Ei})\sin\vartheta_u$$

$$-\,l_{Es}(\bar{z}_{Ei}\vartheta_{Ea}+z_{Ea}\bar{\vartheta}_{Ei})\cos\vartheta_u+i_{mc}^2 y'_{Ea}\bar{y}'_{Ei}+i_{mo}^2(\bar{y}'_{Ei}z'_{Ea}+y'_{Ea}\bar{z}'_{Ei})+i_{ms}^2 z'_{Ea}\bar{z}'_{Ei}\Big]dx$$

$$=\sum_{j=1}^{\infty} q_{ja}\int_0^{R_A} m'\Big[\bar{y}_{Ei}\bar{y}_{Ej}+\bar{z}_{Ei}\bar{z}_{Ej}+i_m^2\bar{\vartheta}_{Ei}\bar{\vartheta}_{Ej}+l_{Es}(\bar{y}_{Ei}\bar{\vartheta}_{Ej}+\bar{y}_{Ej}\bar{\vartheta}_{Ei})\sin\vartheta_u \tag{7.10}$$

$$-\,l_{Es}(\bar{z}_{Ei}\bar{\vartheta}_{Ej}+\bar{z}_{Ej}\bar{\vartheta}_{Ei})\cos\vartheta_u+i_{mc}^2\bar{y}'_{Ei}\bar{y}'_{Ej}+i_{mo}^2(\bar{y}'_{Ei}\bar{z}'_{Ej}+\bar{y}'_{Ej}\bar{z}'_{Ei})+i_{ms}^2\bar{z}'_{Ei}\bar{z}'_{Ej}\Big]dx$$

Because of the orthogonality condition (6.12), all the terms in the sum on the right vanish except for that with $j = i$. Hence, Eq. (7.10) becomes

$$q_{ia}=\frac{1}{N_i}\int_0^{R_A} m'\Big[y_{Ea}\bar{y}_{Ei}+z_{Ea}\bar{z}_{Ei}+i_m^2\vartheta_{Ea}\bar{\vartheta}_{Ei}+l_{Es}(\bar{y}_{Ei}\vartheta_{Ea}+y_{Ea}\bar{\vartheta}_{Ei})\sin\vartheta_u$$

$$-\,l_{Es}(\bar{z}_{Ei}\vartheta_{Ea}+z_{Ea}\bar{\vartheta}_{Ei})\cos\vartheta_u+i_{mc}^2 y'_{Ea}\bar{y}'_{Ei}+i_{mo}^2(\bar{y}'_{Ei}z'_{Ea}+y'_{Ea}\bar{z}'_{Ei})+i_{ms}^2 z'_{Ea}\bar{z}'_{Ei}\Big]dx \tag{7.11}$$

N_j is the denominator constant (generalized mass) from Eq. (6.19). If we replace y_{Ea}, z_{Ea}, θ_{Ea} by $\dot{y}_{Ea}$, $\dot{z}_{Ea}$, and $\dot{\theta}_{Ea}$, we obtain

$$\dot{q}_{ia}=\frac{1}{N_i}\int_0^{R_A} m'\Big[\dot{y}_{Ea}\bar{y}_{Ei}+\dot{z}_{Ea}\bar{z}_{Ei}+i_m^2\dot{\vartheta}_{Ea}\bar{\vartheta}_{Ei}+l_{Es}(\bar{y}_{Ei}\dot{\vartheta}_{Ea}+\dot{y}_{Ea}\bar{\vartheta}_{Ei})\sin\vartheta_u \tag{7.12}$$

$$-\,l_{Es}(\bar{z}_{Ei}\dot{\vartheta}_{Ea}+\dot{z}_{Ea}\bar{\vartheta}_{Ei})\cos\vartheta_u+i_{mc}^2\dot{y}'_{Ea}\bar{y}'_{Ei}+i_{mo}^2(\bar{y}'_{Ei}\dot{z}'_{Ea}+\dot{y}'_{Ea}\bar{z}'_{Ei})+i_{ms}^2\dot{z}'_{Ea}\bar{z}'_{Ei}\Big]dx$$

8. Sample Computations /52

8.1. Programs

In order to be able to calculate the dynamic behavior of rotor blades by the method of coupled flapwise-bending, edgewise bending, and torsion natural modes, a Fortran program was written. It includes all the preceding theory in its most general form, except for the moment of inertia due to expansion of the blade in the transverse direction, which has not yet been incorporated into the program for coupled natural vibrations. Furthermore, aerodynamic calculations are included, so that the program does not need the aerodynamic forces and moments as inputs in addition to the natural vibrations, but instead the rotor geometry, the rotor moduli μ_x and μ_z, and the induced downwash w_i. Other influences on the airflow toward the blade can be incorporated without difficulty.

These mutually compatible programs make it possible to calculate the coupled natural vibrations by the multihinge articulated blade method with a maximum of 40 identical blade segments (i.e. a maximum of 120 degrees of freedom), than to reduce them by a type of selection procedure to a maximum of 20 identical blade segments, and thus to calculate the forced vibrations, e.g. separation flutter. The Runge-Kutta method is employed. Up to 12 degrees of freedom q_i, i.e. 12 natural modes, can be taken, each natural mode consisting of three curves. For the aerodynamic component the theory δ = const. was used from Report 3 of Just and Jaeckel [7]. In addition, however, the wind speeds had to be transformed into the fixed-blade-element system which is wind-tipped because of the blade deformation. Furthermore, the aerodynamic forces occurring on the blade element had to be correspondingly transformed back. Subroutines were called for the unsteady profile coefficients.

The results of the computations are the values of the generalized degrees of freedom q_i as functions of time. Using them, we can calculate the blade retention forces, and, with the aid of the natural moments, the bending and torsion moments together with the associated material stresses, as was done in Amer and LaForge [4]. Furthermore, the behavior of the q_i will indicate whether, in a particular flight situation, the blade vibrations will grow with time, or whether there is stability. Peter Crimi [5] has investigated this question in a more general fashion, although the mathematics is somewhat complicated in that case as well. /53

8.2. Given Data and Curves

As our "experimental animal," we take the Sikorsky S-61 H1 helicopter in the version with flapping hinge but no swivel hinge. Most of the data on the blade is contained in TRECOM T.R. 64-15 [6]. The missing data was chosen so that it was compatible with the given data. The basic parameters are

$$R = 9.45 \text{ m} \qquad a = 0.3206 \text{ m} \qquad l = 0.45$$

$$\omega_{Ro} = 21.24 \text{ s}^{-1} \qquad z = 5 \tag{8.1}$$

The blades have the NACA 0012 profile, and a constant chord of 45 cm, and are thus rectangular. The weight per segment and in particular the rigidity values do vary, however, as shown in the Table on the next page. We will take these weights as being concentrated in a point at the x-position concerned. We choose the following quantities:

$$\begin{array}{ll} E = 7000 \text{ kg/mm}^2 & G = 2700 \text{ kg/mm}^2 \\ m' i_{m\xi}^2 = 18 \cdot 10^{-3} \text{ kg s}^2 & m' i_{m\eta}^2 = 0.756 \cdot 10^{-3} \text{ kg s}^2 \\ i_F = 0.08 \text{ m} & m' l_{ES} = 0.03 \text{ kg s}^2/\text{m} \end{array} \tag{8.2}$$

The latter corresponds to a center of gravity displaced backwards by about 6.5% in comparison with the elastic axis. In the undeformed state, the latter is taken to be a straight line coinciding with the x-axis (see Fig. 2.1). In the following, we calculate with

$$\begin{array}{c} e_A = e_F = B_1 = B_2 = 0 \\ \Delta\vartheta = -4^\circ \qquad \vartheta_c = \vartheta_s = 0 \\ v_x = 73 \frac{m}{s} \qquad v_z = -16{,}9 \frac{m}{s} \qquad \varrho = 0{,}125 \frac{kp\, s^2}{m^4} \end{array} \tag{8.3}$$

which corresponds to a flight speed of v = 75 m/sec and a rotor angle of attack of $\alpha = -13^\circ$. Since the aircraft weight must be about 9300 kg, we require $k_a \approx 0.0133$ in horizontal flight. Using the curves of the DFH Report 42 [8], we obtain /55

$$\vartheta_{a7} \approx 12^\circ \qquad \delta = v_z - wi \sim -19 \frac{m}{s} \tag{8.4}$$

which corresponds to a wi of 2.1 m/xec. We calculate a more precise wi and B from Report 3 [7], Chapter 9, Section 4, or Chapter 3:

TABLE OF BLADE PARAMETERS WHICH ARE FUNCTIONS OF x

$\frac{x}{R-a}$	Segment wt [kg]	Thickness of wall [cm]	I_1 [cm^4]	I_2 [cm^4]	J_T [cm^4]
1.000	1.160	0.183	40.38	420.40	116.50
.950	5.70	0.368	80.33	845.0	228.9
.900	4.31	0.368	81.17	853.3	237.3
.850	4.17	0.371	82.83	857.5	245.6
.800	4.16	0.378	83.25	865.8	249.7
.750	4.21	0.384	84.91	869.9	262.2
.700	4.09_5	0.391	87.41	986.5	274.7
.650	4.14_5	0.401	91.15	999.0	283.0
.600	4.28_6		95.32	1011.5	291.4
.550	4.25		97.82	1032.3	303.8
.500	4.49_4	0.401	101.98	1123.8	316.3
.450	4.51	0.457	104.48	1136.3	328.8
.400	4.54		108.22	1165.5	341.3
.350	4.41		112.80	1186.3	353.8
.300	3.98	0.457	116.55	1219.6	374.6
.250	4.11	0.513	121.13	1240.4	412.1
.200	3.87	0.533	126.54	1269.5	582.7
.150	4.77	0.673	183.15	1456.8	1248.7
.100	25.04	2.540	1873.1	2913.7	2081.2
.050	38.17	5.080	3121.8	3121.8	4162.4
.03394	9.280	5.080	3121.80	3121.80	4162.40

$$wi \approx 1.77 \text{ m/s} \qquad B \approx 0.984 \qquad (8.5)$$

From all the previously given data with the exception of l, θ_c, θ_s, v_x, v_z, ζ, wi and B, we obtain the natural vibrations. In the sense of the natural-mode method, they are a part of the input data, but, as a whole, are intermediate results, and are therefore presented in Chapter 8.3.

The assumed initial state for the forced vibrations can be described by

$$\psi_{B1}(t = 0) = 0 \qquad \left[y_E \; z_E \; \vartheta_E \; \dot{y}_E \; \dot{z}_E \; \dot{\vartheta}_E\right](t = 0) \equiv 0 \qquad (8.6)$$

The "$\equiv$" means "for all x." It is true that, in the sense of Chapter 7, any arbitrary initial deflection curves could be used as input, but in this as in other assumptions, the objective is to organize the numerical examples to be as simple and clear as possible.

Lastly, the $[c_a \; c_w \; c_m]$ (α_{eff}) curves employed are important output data. For the numerical examples of this report, we used subroutines based on the unsteady coefficient curves depicted in Fig. 8.1. Various curves valid for the NACA 0012 profile were averaged and schematicized. The coefficients depend not only on α_{eff} but also on the absolute value of $\dot{\alpha}_{eff}$. More precise subroutines, in particular allowing for the Mach-number effect, are naturally desirable for practical calculations, and the Institute already has some of them. For our rather illustrative examples, these simple subroutines are sufficient, however. One important reason is that the largest and most influential aerodynamic forces were calculated for small positive values of α_{eff}. /56

8.3. Results of Computation and Discussion /57

Four examples were calculated with the output data presented in the preceding chapter. The examples differ both in the natural modes permitted as degrees of freedom, and in the fact that the x-y-z coordinate system matches the rotation through $\beta_{B1}(t)$ in the sense of Chapter 6 in one case, and does not in the other. The situation is summarized in the following table.

Example	NB	Natural modes	NU	See Figs.
I	0	1. M_β	2	8.6 - 8.7
II	1	1. M_β 2. M_β | 1. M_ζ	2	8.8 - 8.9
III	0	1. M_β 2. M_β | 1. M_ζ	3	8.10 - 8.15
IV	1	1. M_β 2. M_β | 1. M_ζ | 1. M_τ	2	8.16 - 8.19

NU = number of rotations about rotor axis

$1.M_\beta$ $(2.M_\beta)$ = First (second) flapwise natural mode

$1.M_\zeta$ = " edgewise natural mode

$1.M_t$ = " torsion natural mode

NB = 0(1): the coordinate system is (is not) rotated through the previously calculated flapping angle $\beta_{B1}(t)$.

This means that for NB = 0, apart from the natural modes mentioned, there is also the degree of freedom "rotation about the flapping axis," which we will call $0.M_\beta$. The associated q, i.e. $q(0.M_\beta)$, is calculated just like $q(1.M_\beta)$, but uncoupled and starting from a nonrotated system. Then the other q's are calculated, relative to the system rotated by $\beta_{B1}(t)$, where β_{B1}, $\dot{\beta}_{B1}$ and $\ddot{\beta}_{B1}$ are easily obtained from q, $\dot{q}$, and $\ddot{q}$ of $0.M_\beta$. It is sufficient to calculate $\beta_{B1}(t)$ just one Runge-Kutta step in advance, and then to bring along the remaining q. They are naturally affected by $q(0.M_\beta)$, particularly $q(1.M_\beta)$, which almost vanishes for NB = 0.

The coupled natural modes used in the example are depicted in Figs. 8.1 through 8.4. The associated natural frequencies are /58

Natural mode	$1.M_\beta$	$2.M_\beta$	$1.M_\zeta$	$1.M_\tau$
Natural frequency (angular frequency) in sec^{-1}	21.813	56.18	18.02	144.6

(8.7)

Forty segments were used in calculating the natural vibrations, and this number was reduced to ten in order to calculate the forced vibrations. A reduction to 20 segments would also have been possible. Calculating the generalized aerodynamic forces for forced vibrations (see Eq. (6.19)) naturally requires fewer segments than calculating the natural vibrations. The natural modes were determined so that the principal deflection curve in each case has the value unity at Point 41, and thus is somewhat less than unity at the tip of the blade, which coincides with Point 40. The deflections in Figs. 8.1 through 8.4 have the following dimensions:

Deflection	y_E	z_E	ϑ_E
Dimension	[m]	[m]	[^]

(8.8)

The degrees of freedom q_i in Figs. 8.6 through 8.19 are thus dimensionless.

Example I shows $q(0.M_\beta)$ and $q(1.M_\beta)$ or, expressed somewhat differently, $q(1.M_\beta)$, first in the stationary system, and then in the system rotated by $\beta_{B1}(t)$, where the latter q, as it must, almost vanishes. The building-up process is virtually complete by the end of the first rotation, and so the system is very stable.

In example II, the first two flapwise natural modes and the first edgewise natural mode are taken into account, relative to the ($\beta_{B1} \equiv 0$)-system. The weak $q(2.M_\beta)$ and the strong $q(1.M_\zeta)$ are striking. Further details are provided by the discussion of the next example.

Example III shows the same degrees of freedom, but in the "flapping" coordinate system. Initially, the statement made about the q-curves in examples I and II also hold in this case. However, the $q(1.M_\zeta)$ curve eventually acquired almost exactly the opposite deflection direction as in Example II. This is due to the β_{B1}-$\dot{\beta}_{B1}$ Coriolis forces, which, in the first rotation, particularly between $\psi = 80°$ and $\psi = 180°$ impart to the blade a strong impulse in the forward direction and thus impose upon it a different motion behavior. Edgewise bending can thus be calculated correctly only with NB = 0. /59

The end of the building-up process in $q(1.M_\zeta)$ cannot be foreseen even after three rotations. This means that, under the assumed conditions, perturbations would continue to act, and even be amplified, as long as a skilled pilot did not counteract them. This circumstance is closely related to the small difference between ω_{R_0} and the first edgewise natural frequency. The installation of a swivel hinge, which is indeed normally present, or detuning the first edgewise natural frequency in another fashion would be advantageous in this case.

In Example IV, in which the step width was reduced from $\Delta\psi_{B1} = 10°$ to $\Delta\psi_{B1} = 5°$, with the result that the $\Delta\psi_{B1} = 10°$ computation was found to be quite usable, we have the degree of freedom $1.M_T$, i.e. the first torsion natural mode, in addition to the degrees of freedom $1.M_\beta$, $2.M_\beta$, and $1.M_\zeta$. They bring about more fundamental changes in the situation than $0.M_\beta$ does in Example III. The latter variable is again omitted here, in order to study the influences separately. The most conspicuous phenomenon is the amplification of $q(1.M_T)$. As a rough calculation shows, it is futile to look for the explanation in the hysteresis loops of the $c_a(\alpha_{eff})$ and the $c_m(\alpha_{eff})$ curves, although both of them could conceivably contribute to amplifying torsion, the former in connection with a positive lever arm for the lift force. The situation becomes clear if one considers the interaction between $q(1.M_\beta)$ and $q(1.M_T)$. If $q(1.M_T)$ is large, this makes the blade angle of attack, the lift and thus $\ddot{q}(1.M_\beta)$ particularly large. From this point of view, $q(1.M_\beta)$ and $q(1.M_T)$ must therefore vibrate in countermotion. The up-and-down motion /60
of the blade now induces a further component in the lift, one proportional to $-\dot{q}(1.M_\beta)$, since the downward flapping of the blade increases the lift. This secondary lift component always reaches its maximum when $\dot{q}(1.M_T)$ is greatest, so that it feeds power to $1.M_T$ when it induces a buckling moment.

This amplification effect, which can be provided with a more exact mathematical foundation, therefore rests on the fact that a lifting force induces a buckling moment. This occurs in our example because we apply the lift force to the x-axis, while the center of gravity of the blade is further back, since $l_{ES} \approx 6\% l > 0$ (cf. Fig. 2.1 and Chapter 8.1). In the first torsion natural mode, this circumstance is expressed in the positive z_E-component (see Fig. 8.5). In the first torsion natural vibration, the blade elements therefore do not rotate about the elastic axis, but roughly about the center-of-gravity axis.

In the first rotation (see Figs. 8.16 and 8.17), the buckling lift moment, which occurs not only dynamically, but also statically due to the centrifugal forces when z_E' is positive, causes increases in the blade angle of attack, and thus in the thrust and in $q(1.M_\beta)$ as well in comparison with Example II. In the second rotations, however, the torsion vibration is already so strong that α_{eff} fluctuates constantly between large positive (greater than 16°) and negative values, so that the mean value of $q(1.M_\beta)$ and thus the rotor thrust become zero. The torsion is further amplified by repeated circuits of the c_a-α_{eff} hysteresis loop. Naturally, in this case, the blade experiences a strong constant drag and an even stronger sin ψ_{B1}-shaped drag, which is reflected in the behavior of $q(1.M_\zeta)$.

It is unlikely that the Sikorsky S-61 blade is so unfavorably /61
designed that all this can really happen. Instead, the center-of-

gravity axis, as opposed to our assumption, will be so far forward that the lift force no longer has a buckling moment relative to the blade center of gravity. Hence, the computation does not demonstrate the existence of a faulty design, but merely indicates the need for caution.

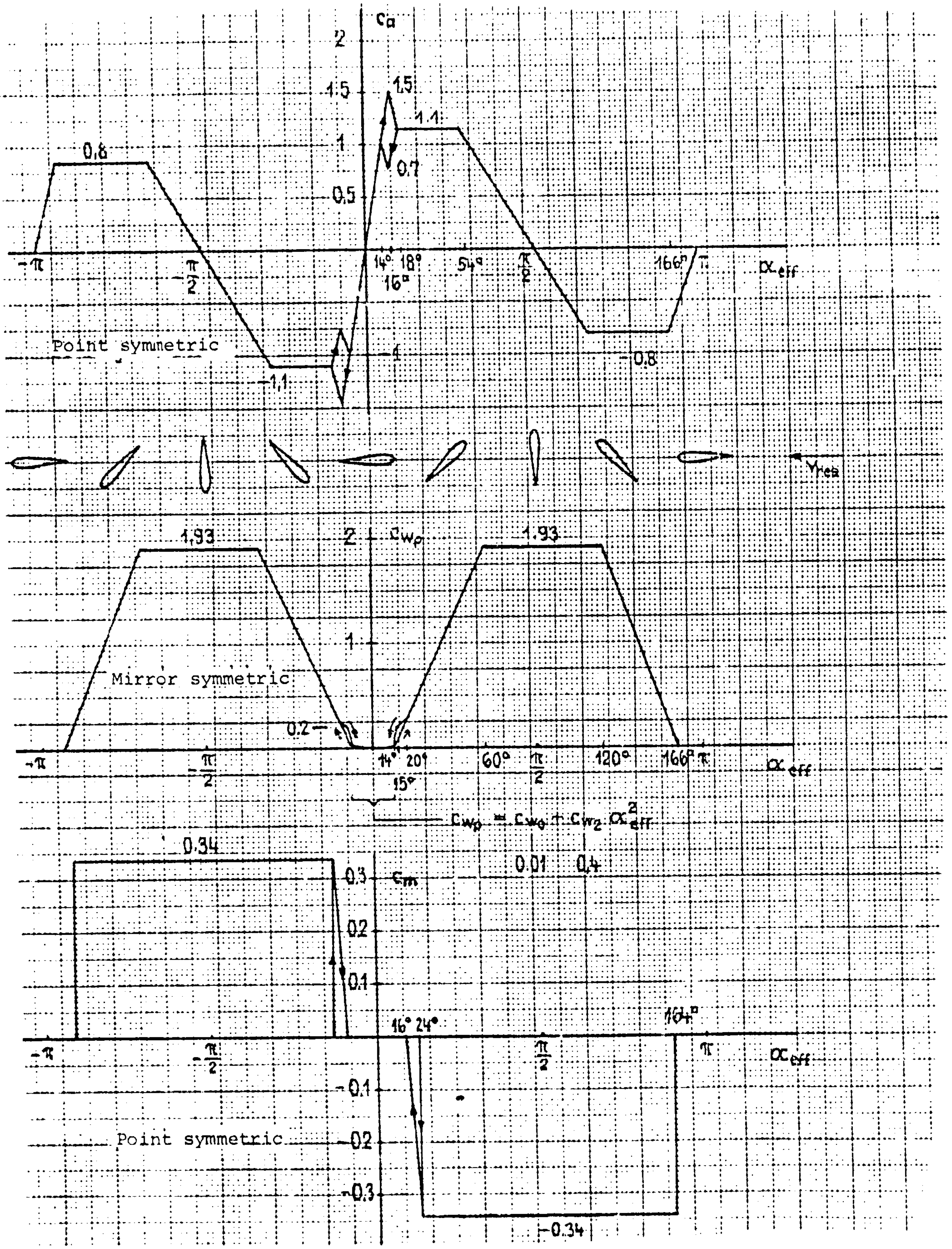

Fig. 8.1. Schematics of unsteady profile coefficients.

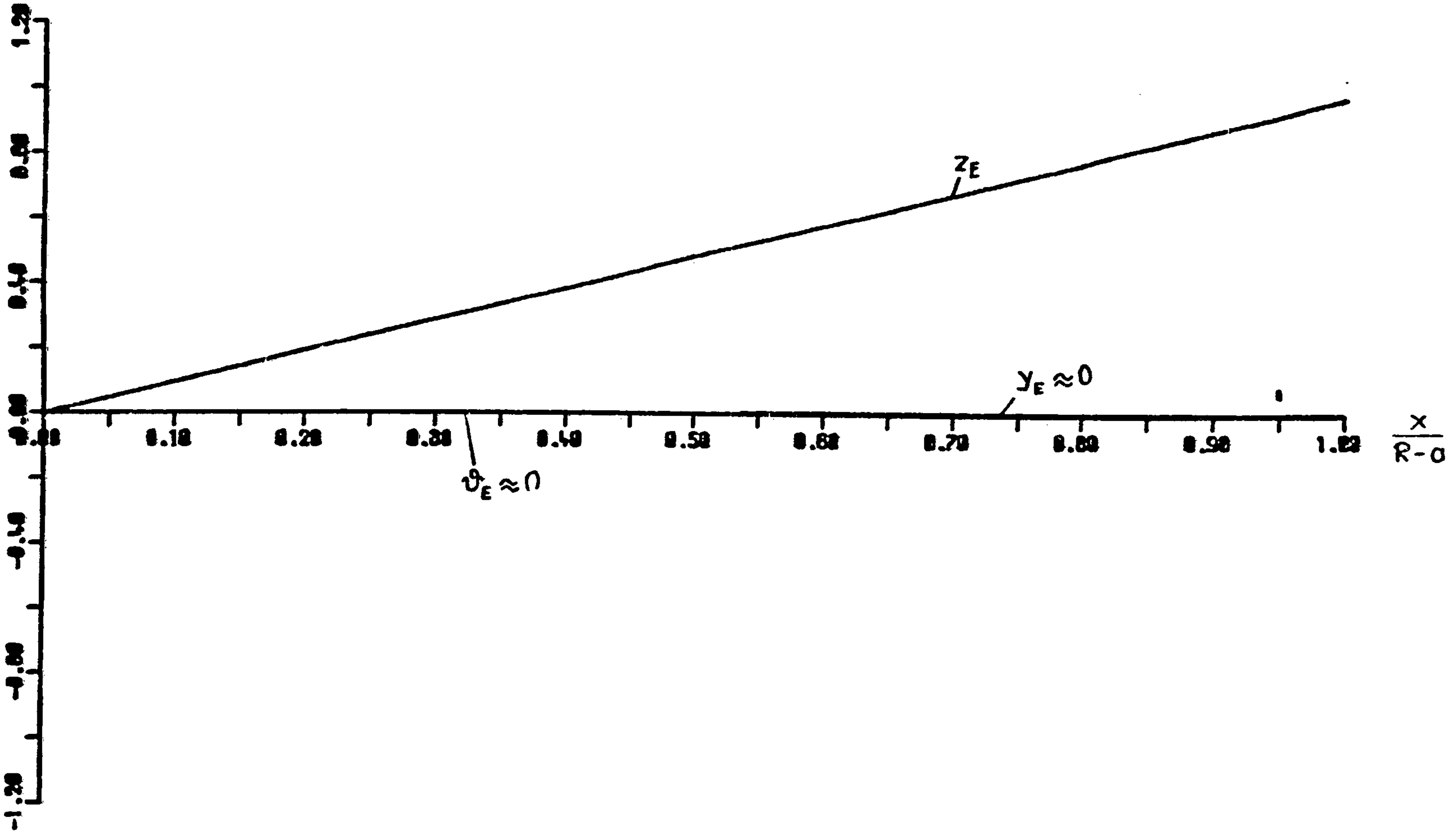

Fig. 8.2. First flapwise natural vibration.

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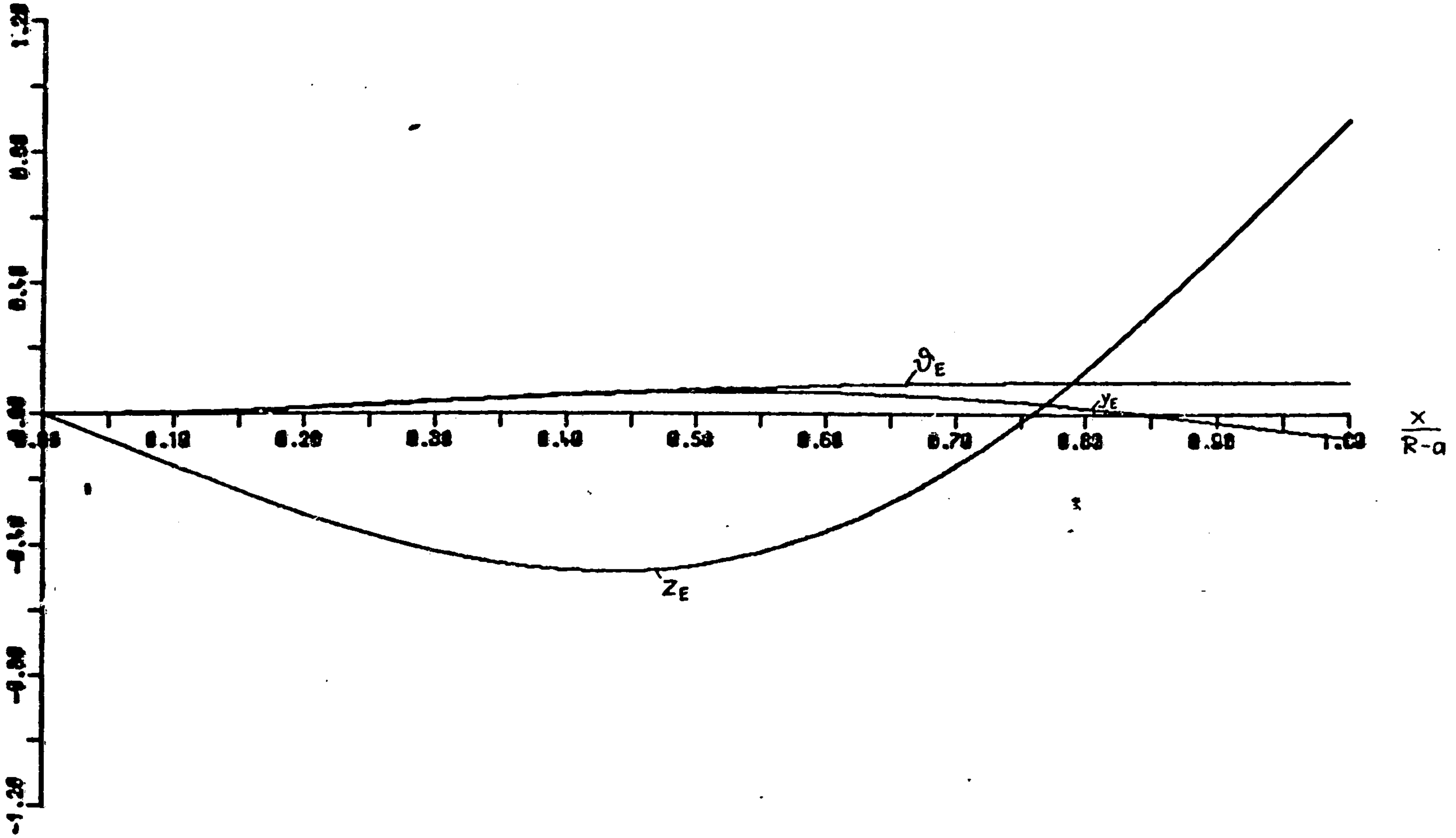

Fig. 8.3. Second flapwise natural vibration.

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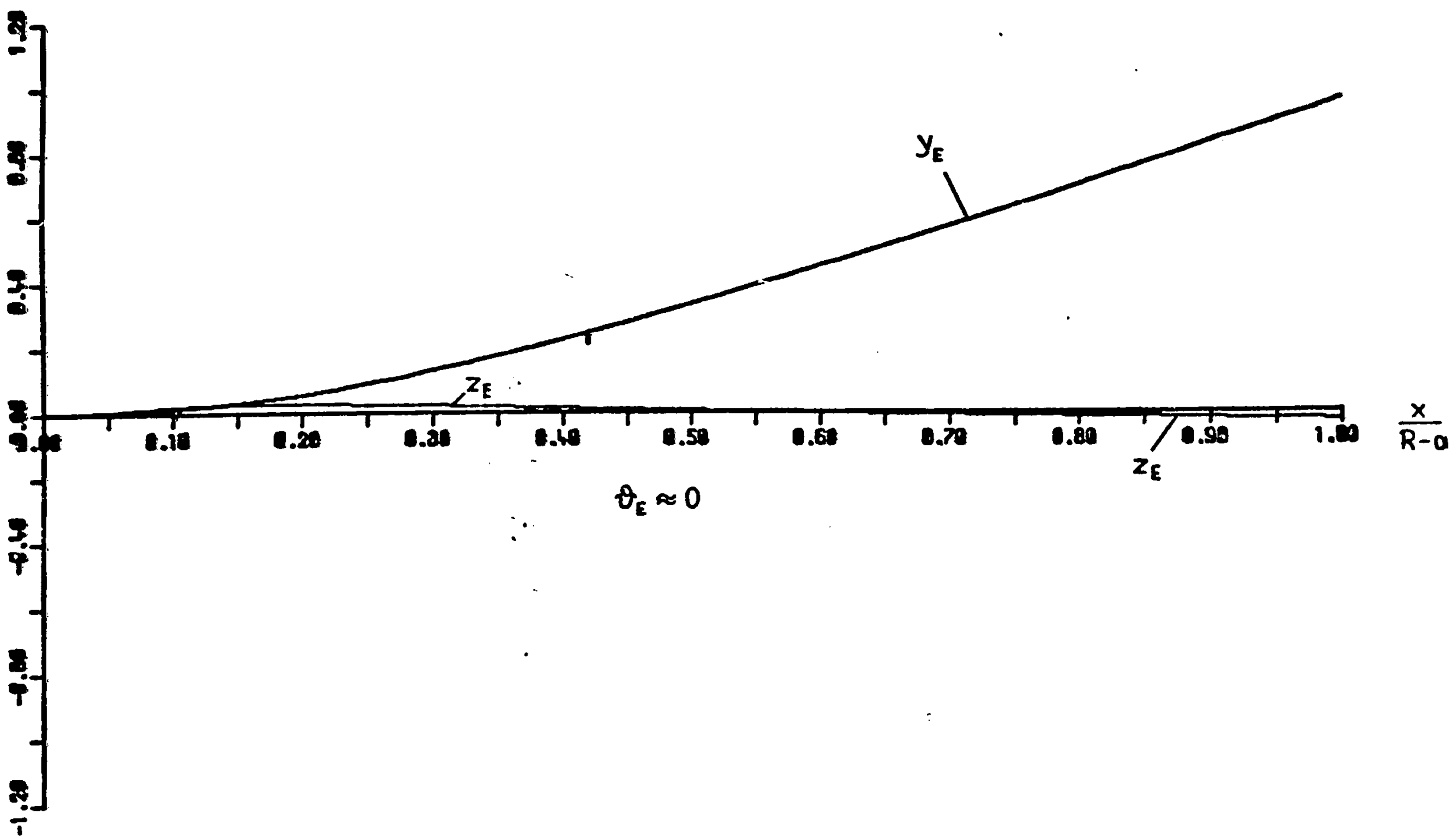

Fig. 8.4. First edgewise natural vibration.

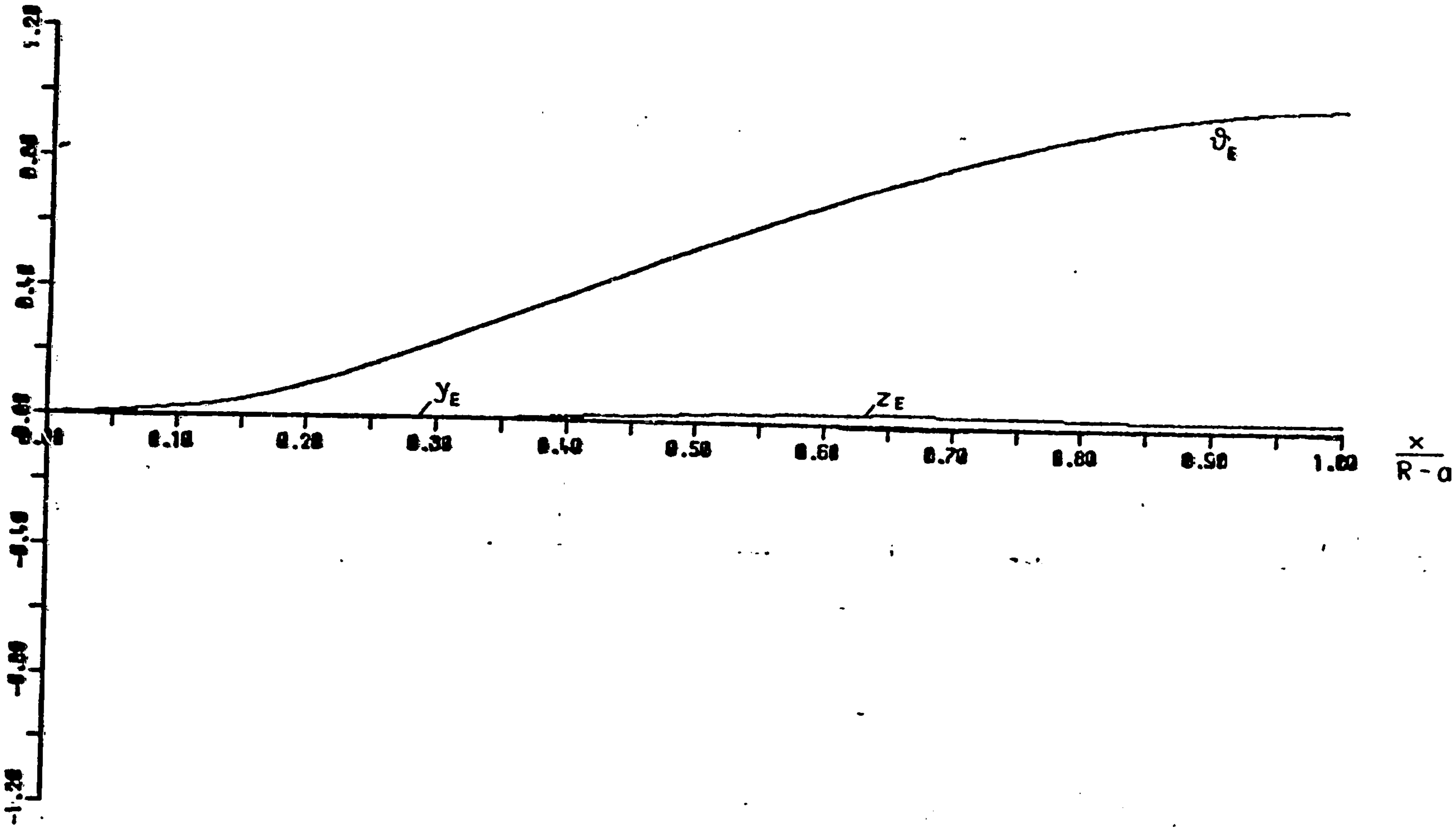

Fig. 8.5. First torsion natural vibration.

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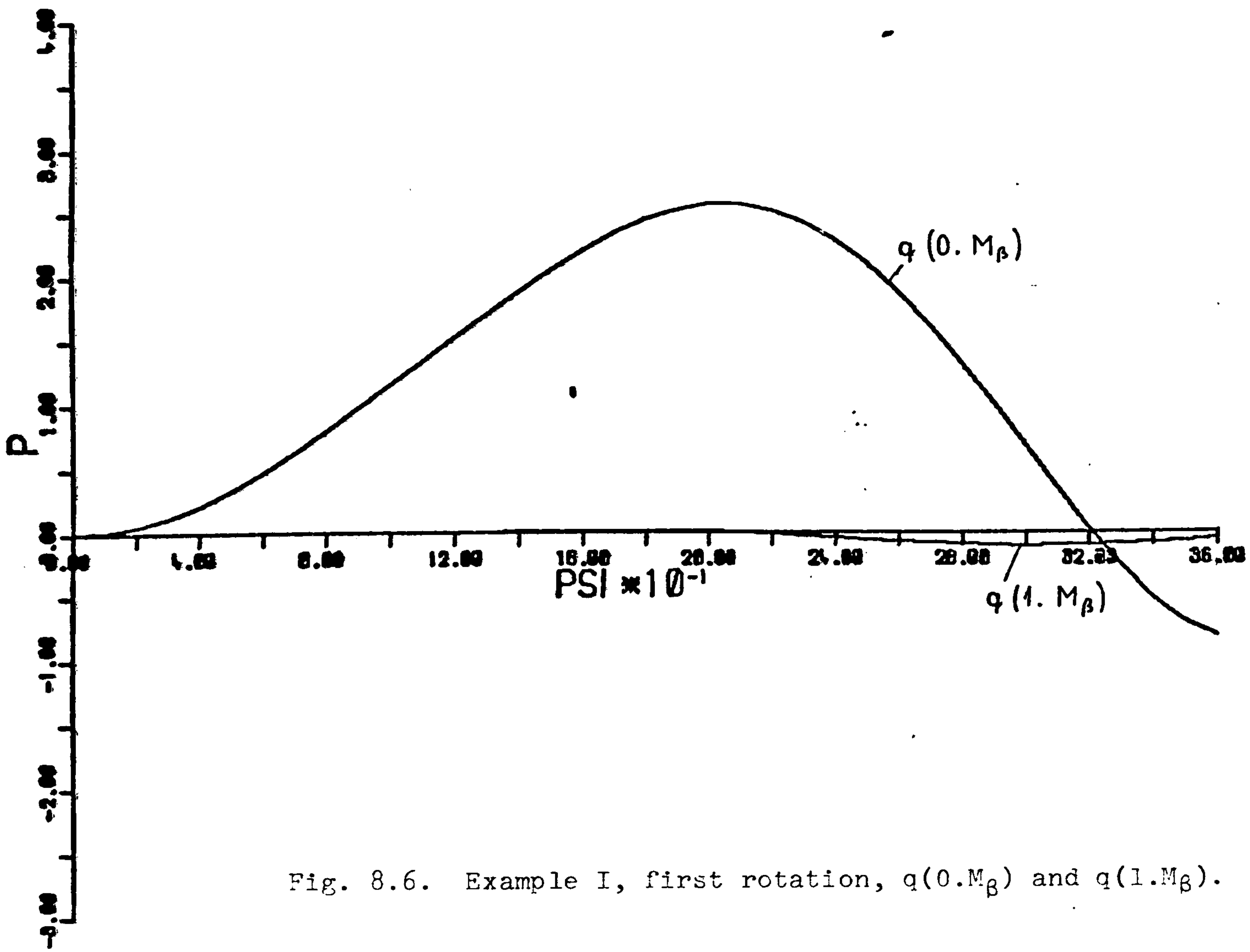

Fig. 8.6. Example I, first rotation, $q(0.M_\beta)$ and $q(1.M_\beta)$.

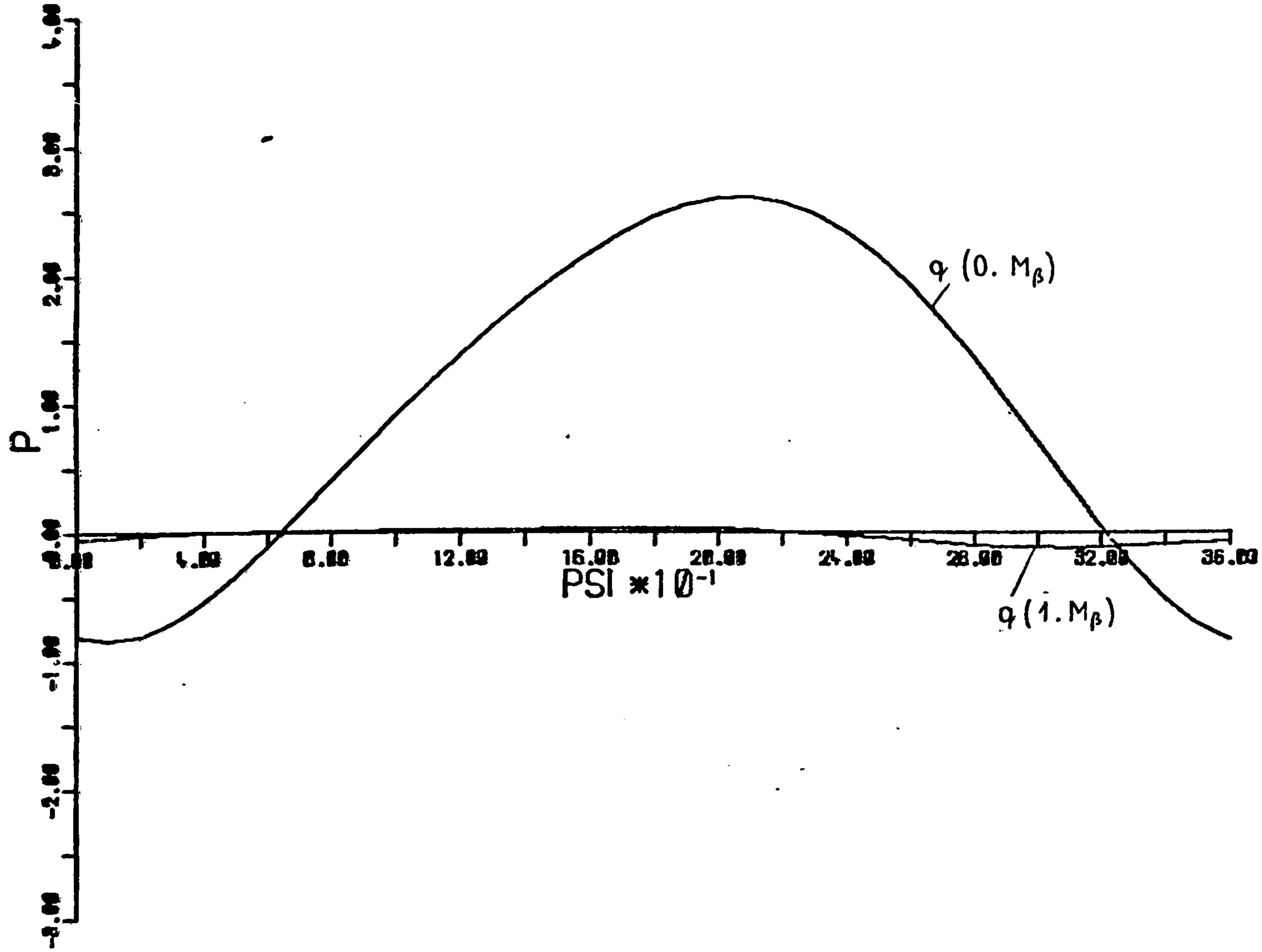

Fig. 8.7. Example I, second rotation, $q(0.M_{\beta})$ and $q(1.M_{\beta})$.

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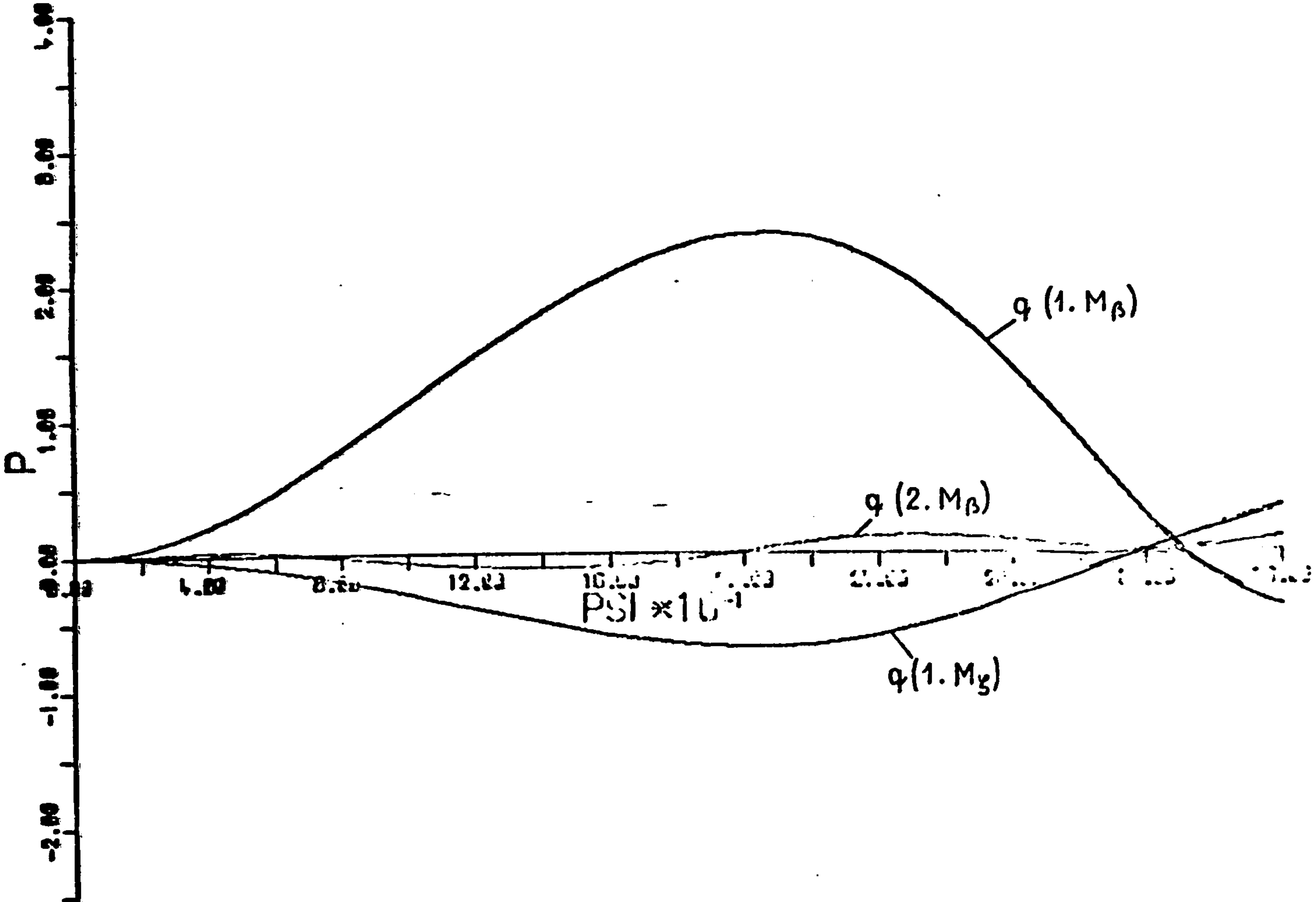

Fig. 8.8. Example II, first rotation, $q(1.M_\beta)$, $q(2.M_\beta)$, and $q(1.M_\zeta)$.

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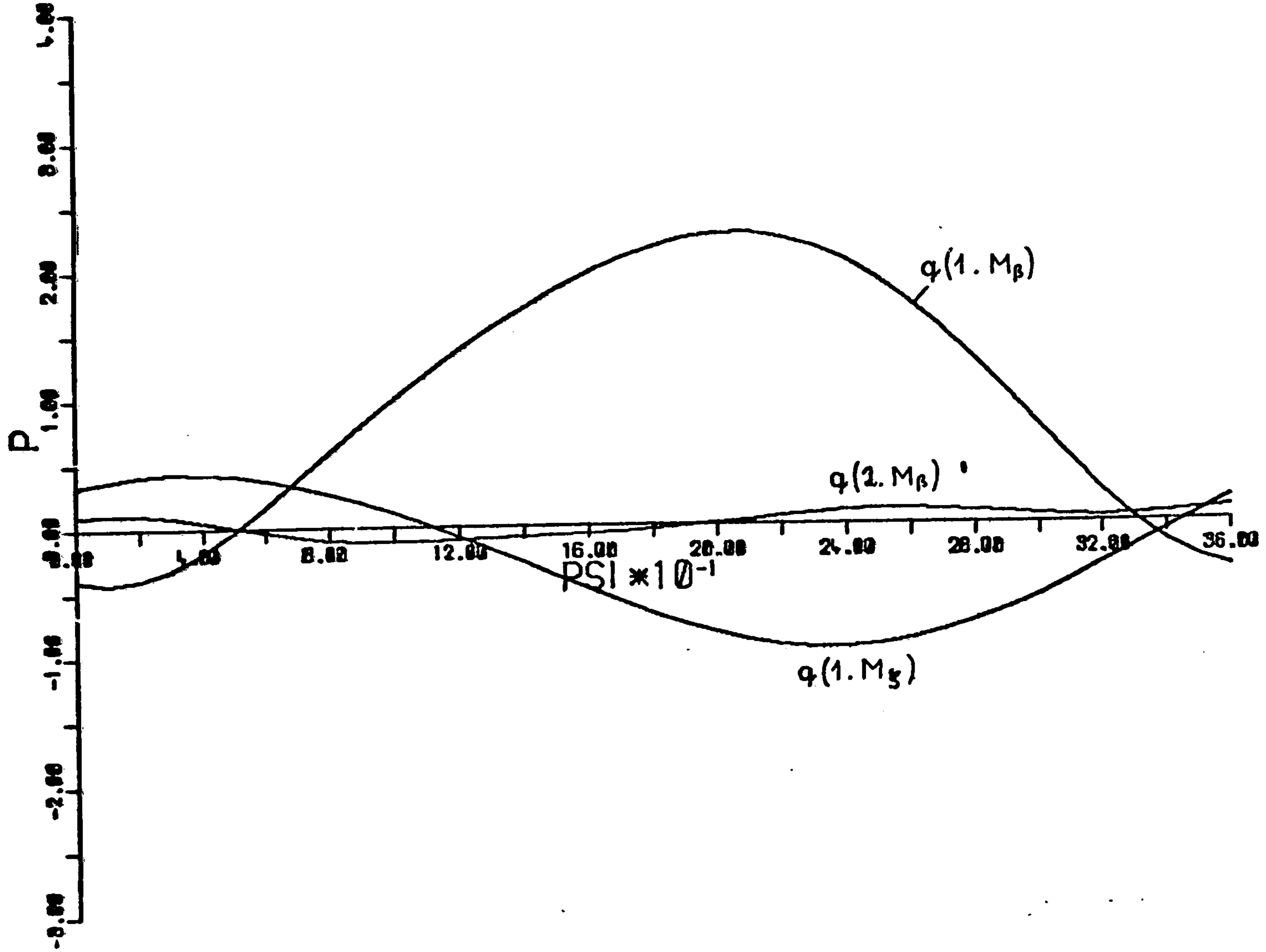

Fig. 8.9. Example, II, second rotation, $q(1.M_\beta)$, $q(2.M_\beta)$ and $q(1.M_\zeta)$.

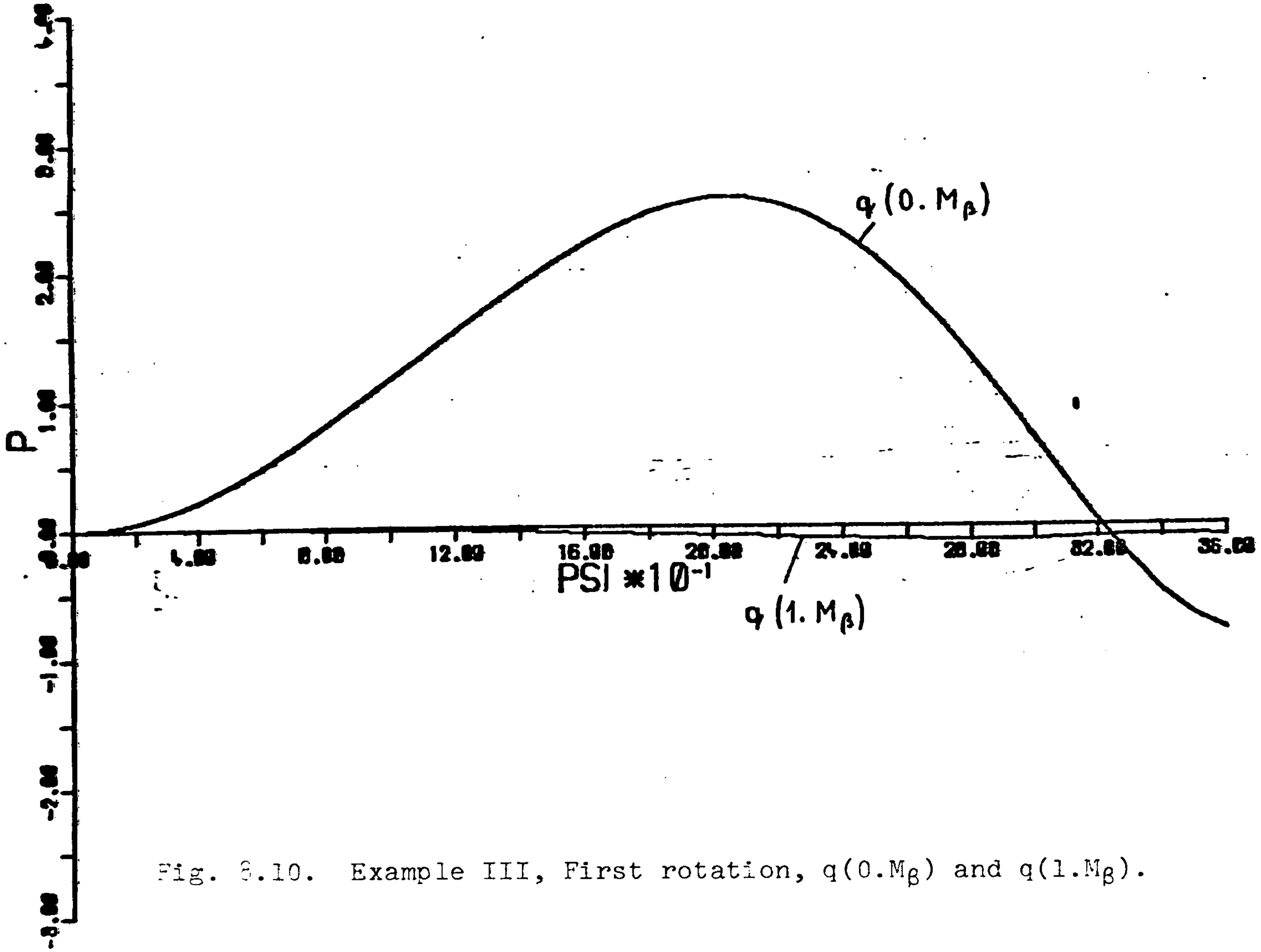

Fig. 8.10. Example III, First rotation, $q(0.M_\beta)$ and $q(1.M_\beta)$.

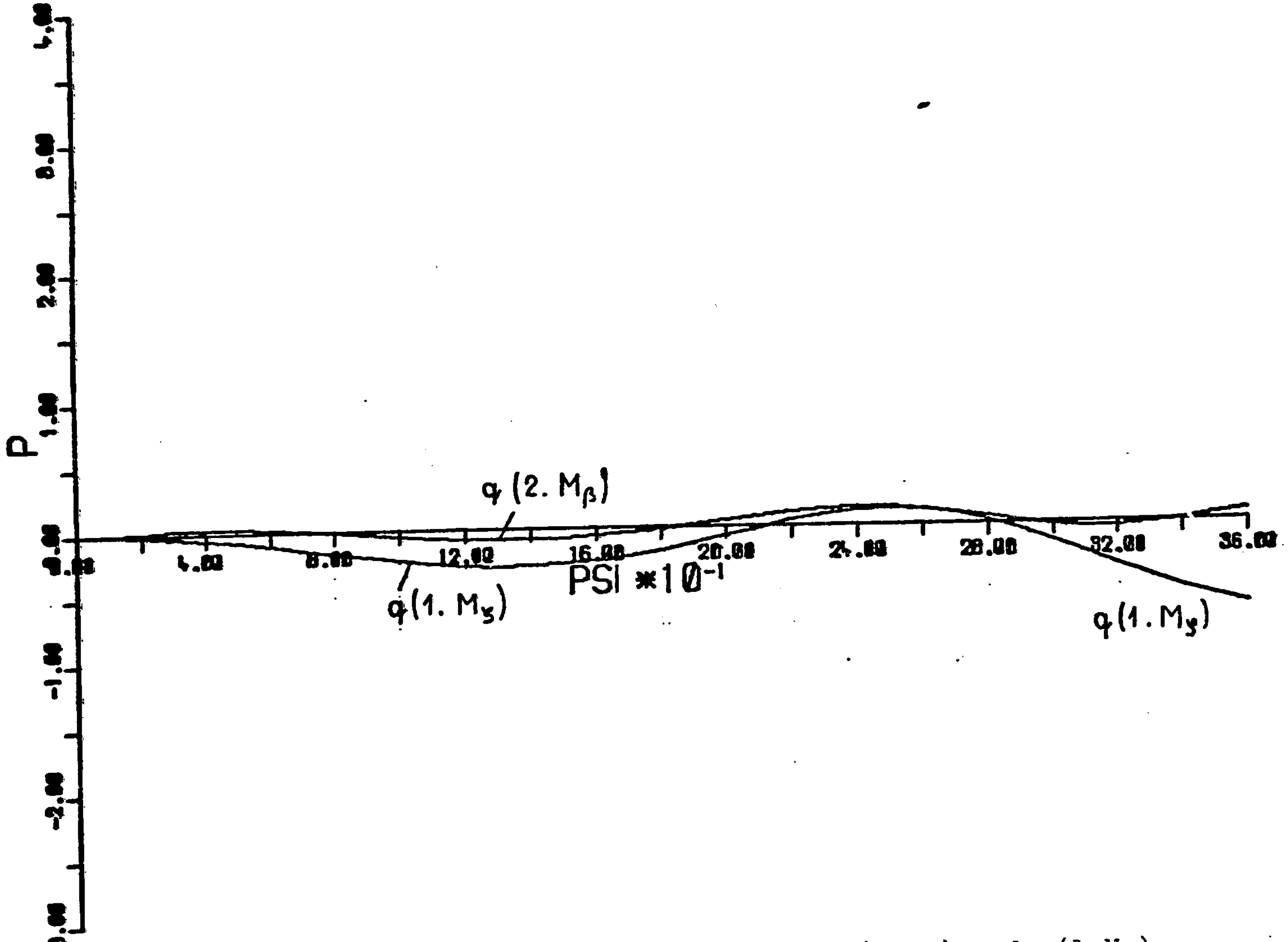

Fig. 8.11. Example III, first rotation, $q(2.M_\beta)$ and $q(1.M_\zeta)$.

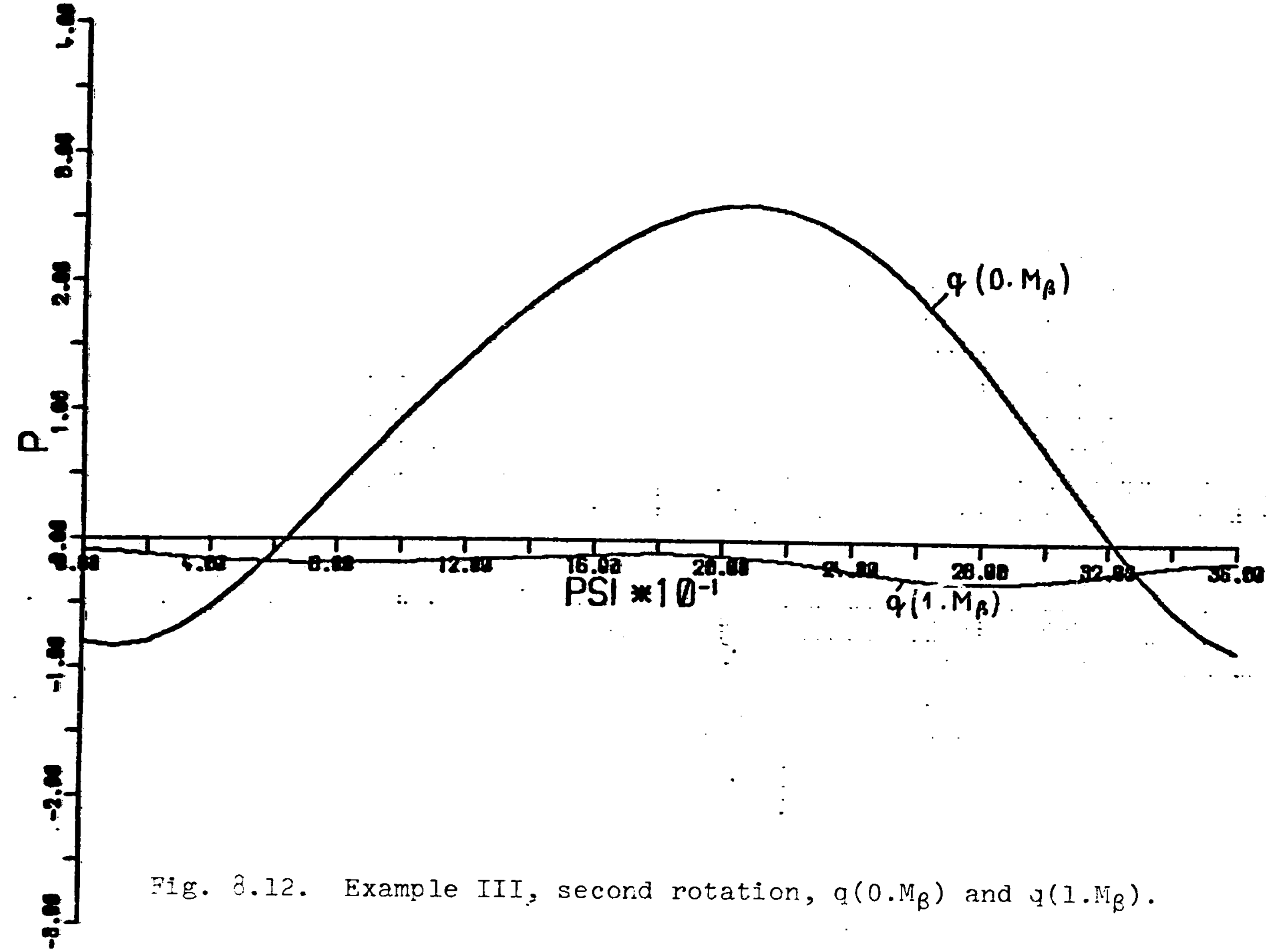

Fig. 8.12. Example III, second rotation, $q(0.M_\beta)$ and $q(1.M_\beta)$.

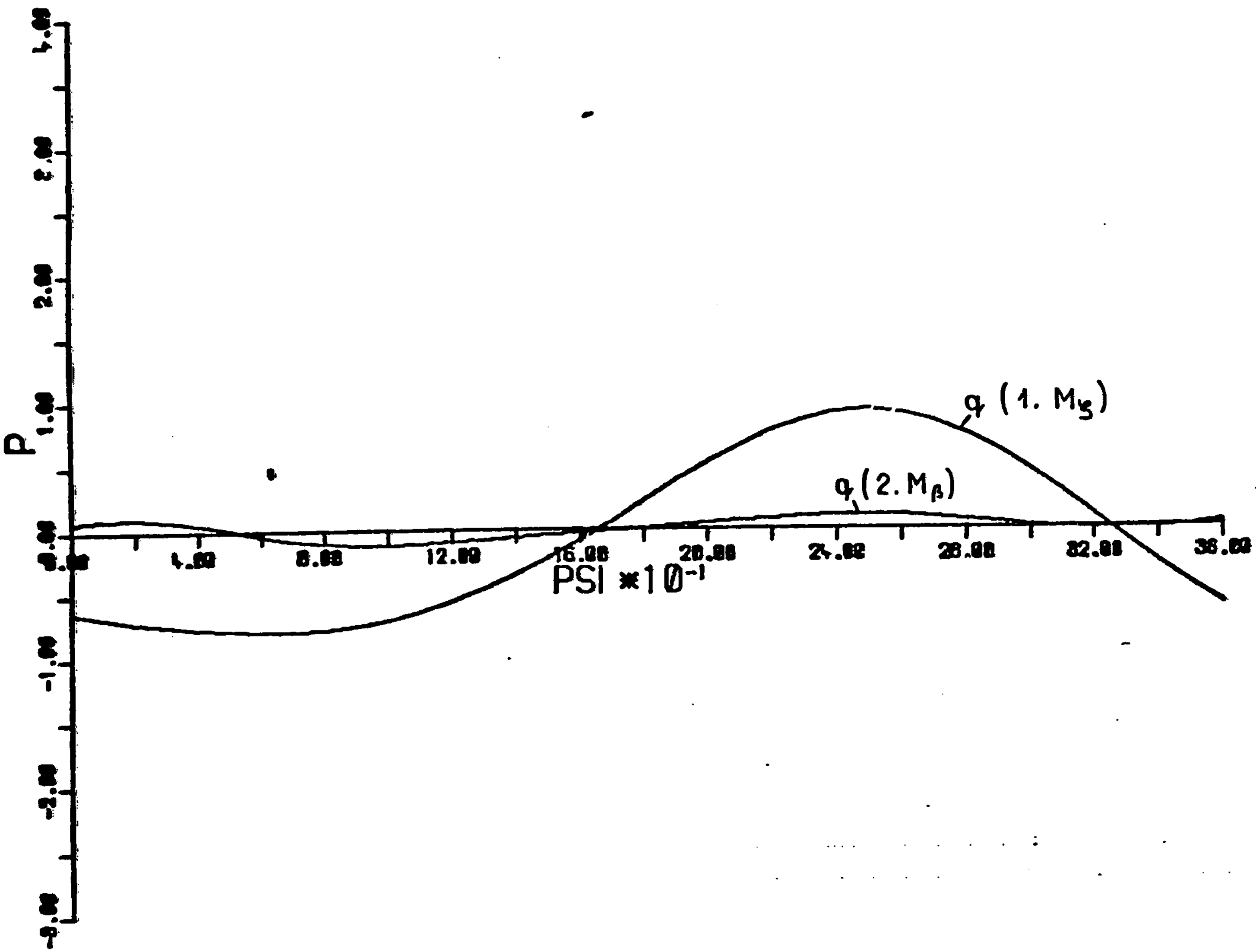

Fig. 3.13. Example III, second rotation, $q(2.M_\beta)$ and $q(1.M_\zeta)$.

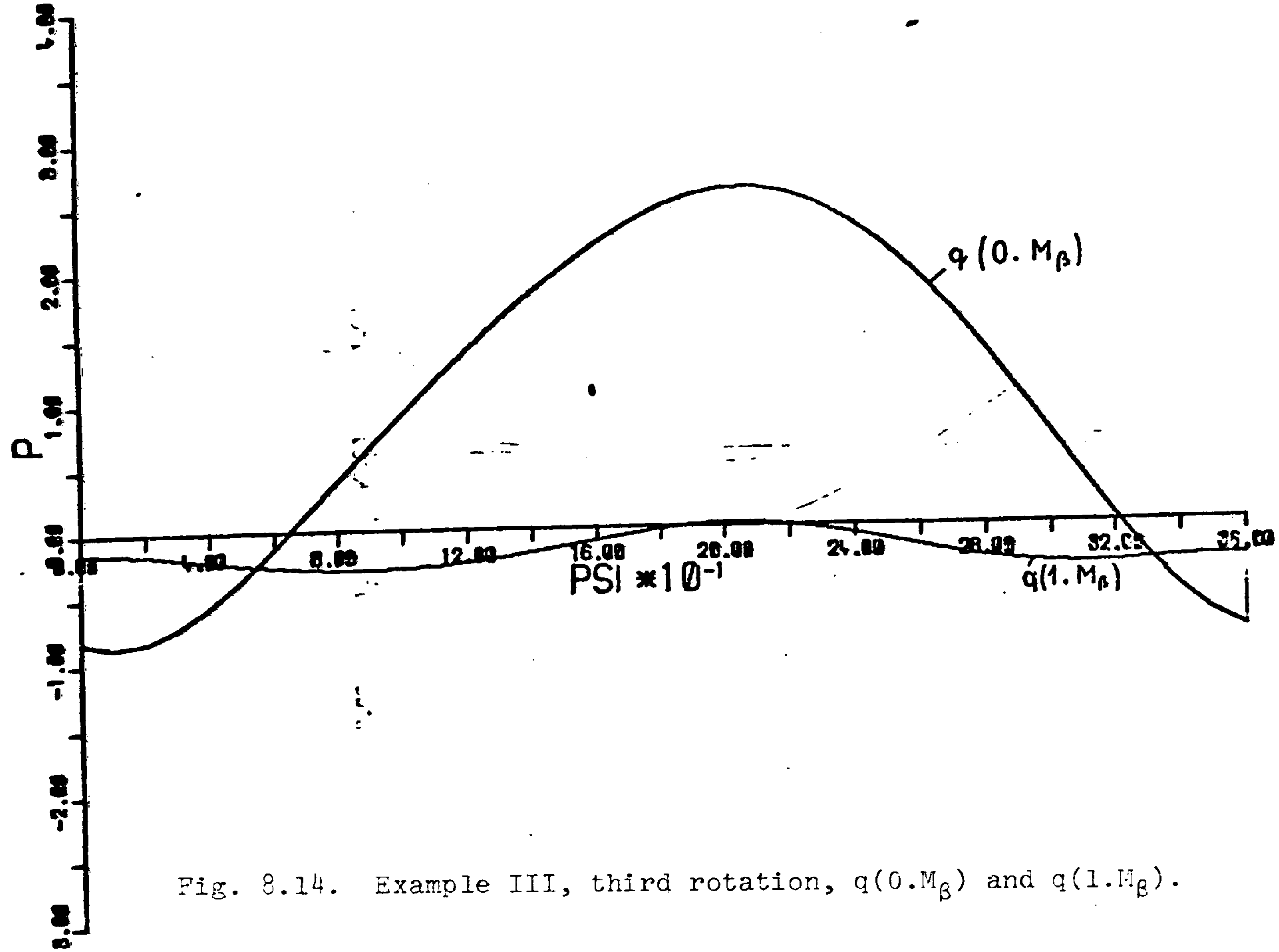

Fig. 8.14. Example III, third rotation, $q(0.M_\beta)$ and $q(1.M_\beta)$.

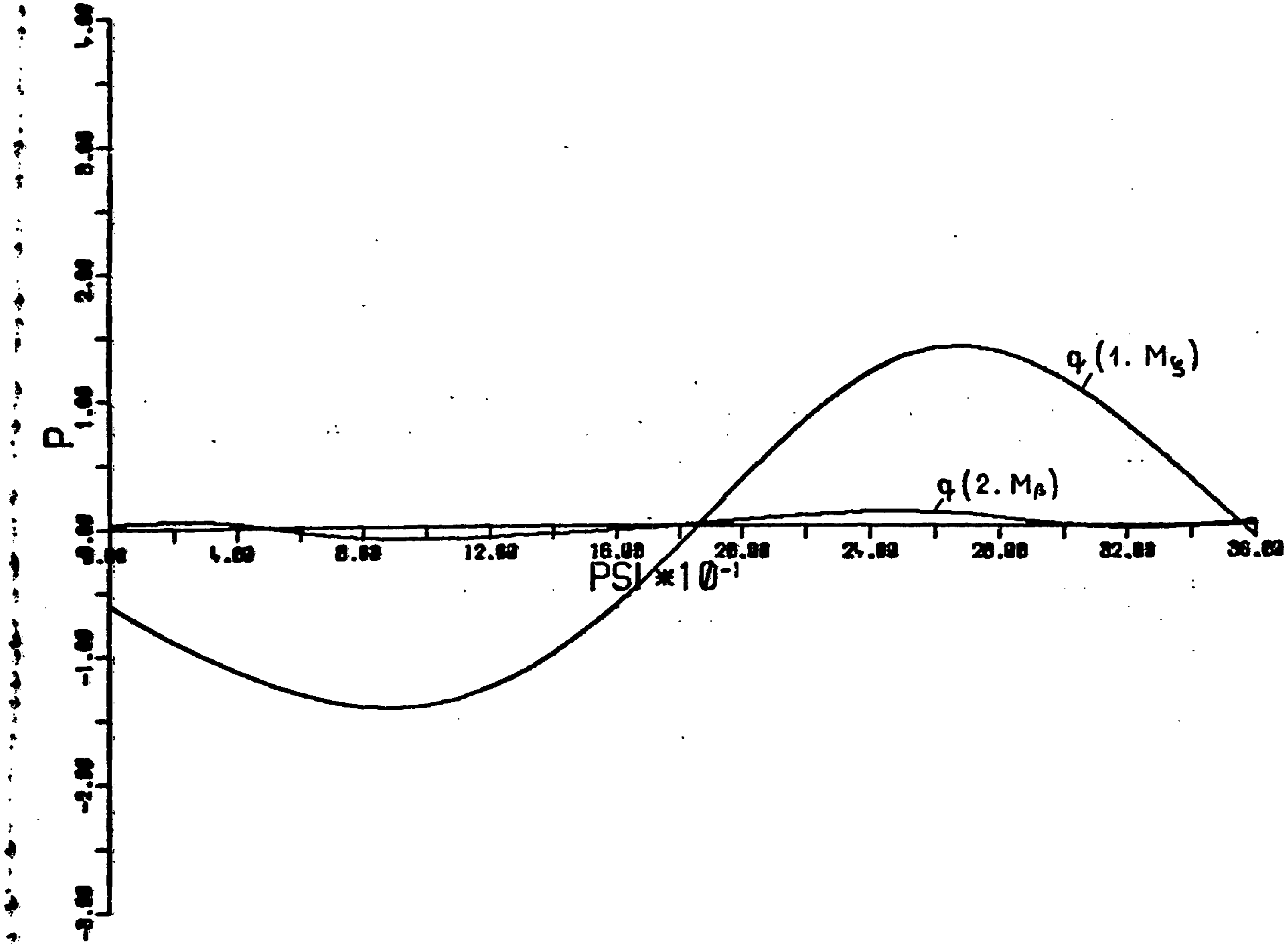

Fig. 8.15. Example III, third rotation, $q(2.M_\beta)$ and $q(1.M_\zeta)$.

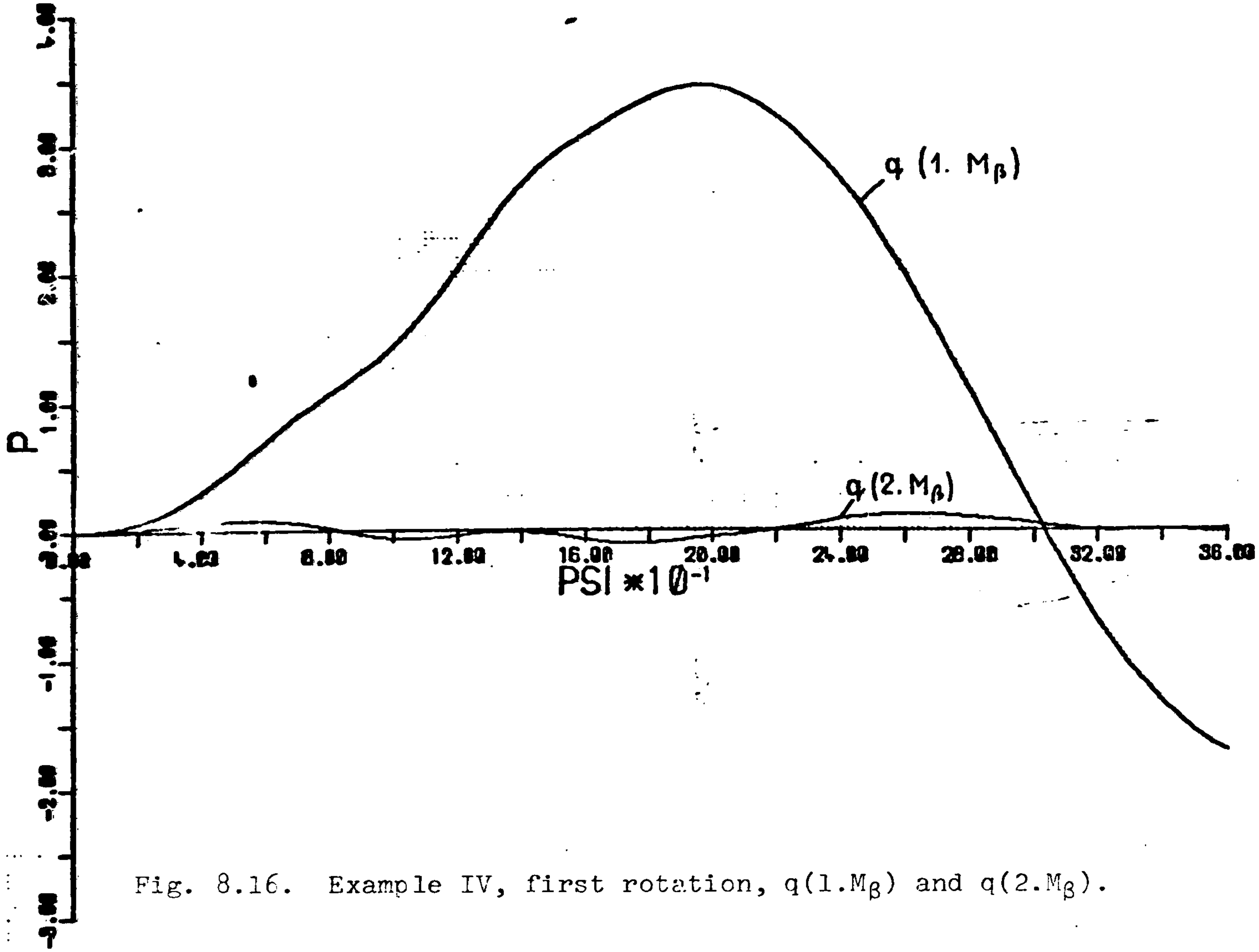

Fig. 8.16. Example IV, first rotation, $q(1.M_\beta)$ and $q(2.M_\beta)$.

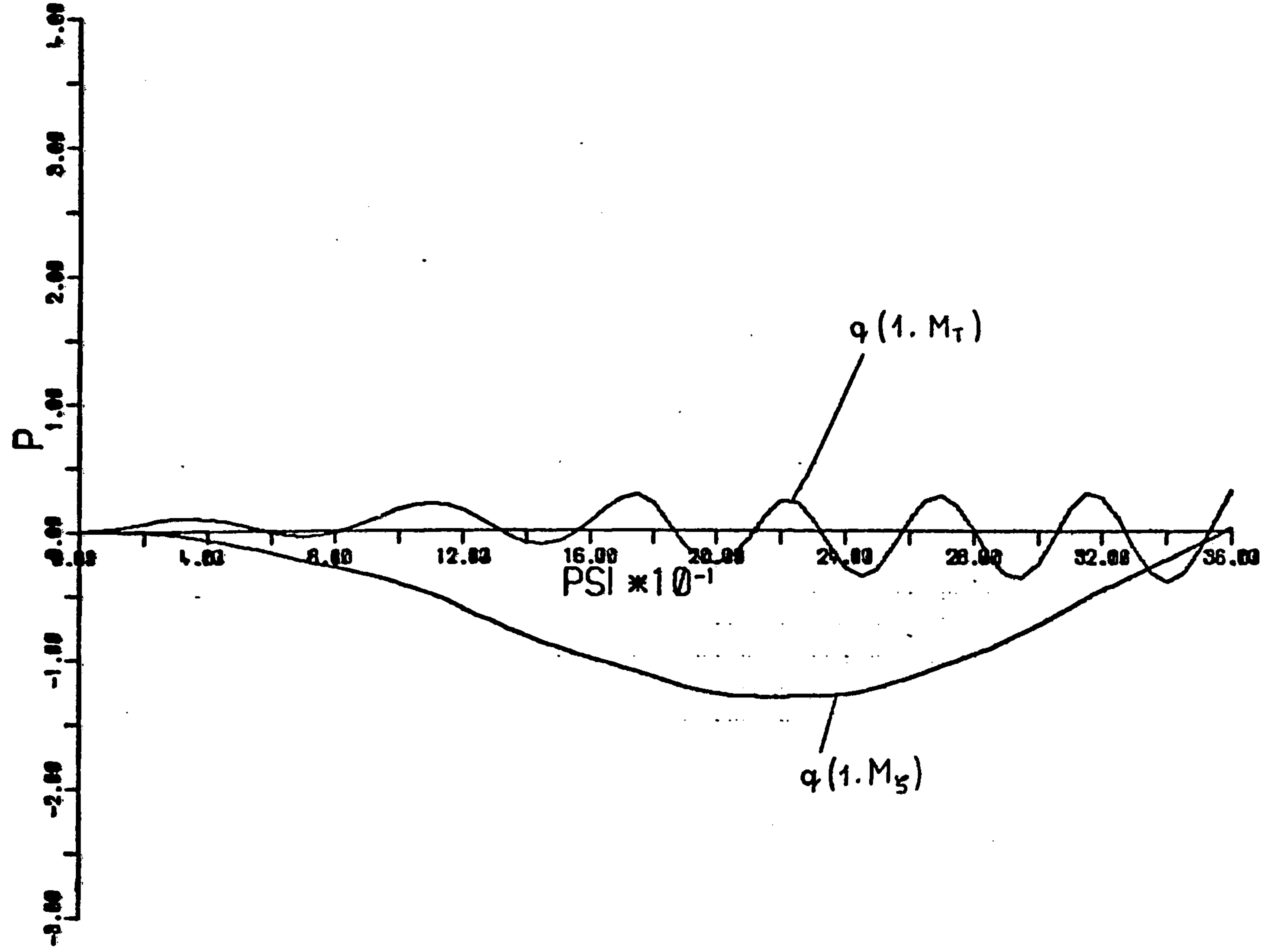

Fig. 8.17. Example IV, first rotation, $q(1.M_\zeta)$ and $q(1.M_T)$.

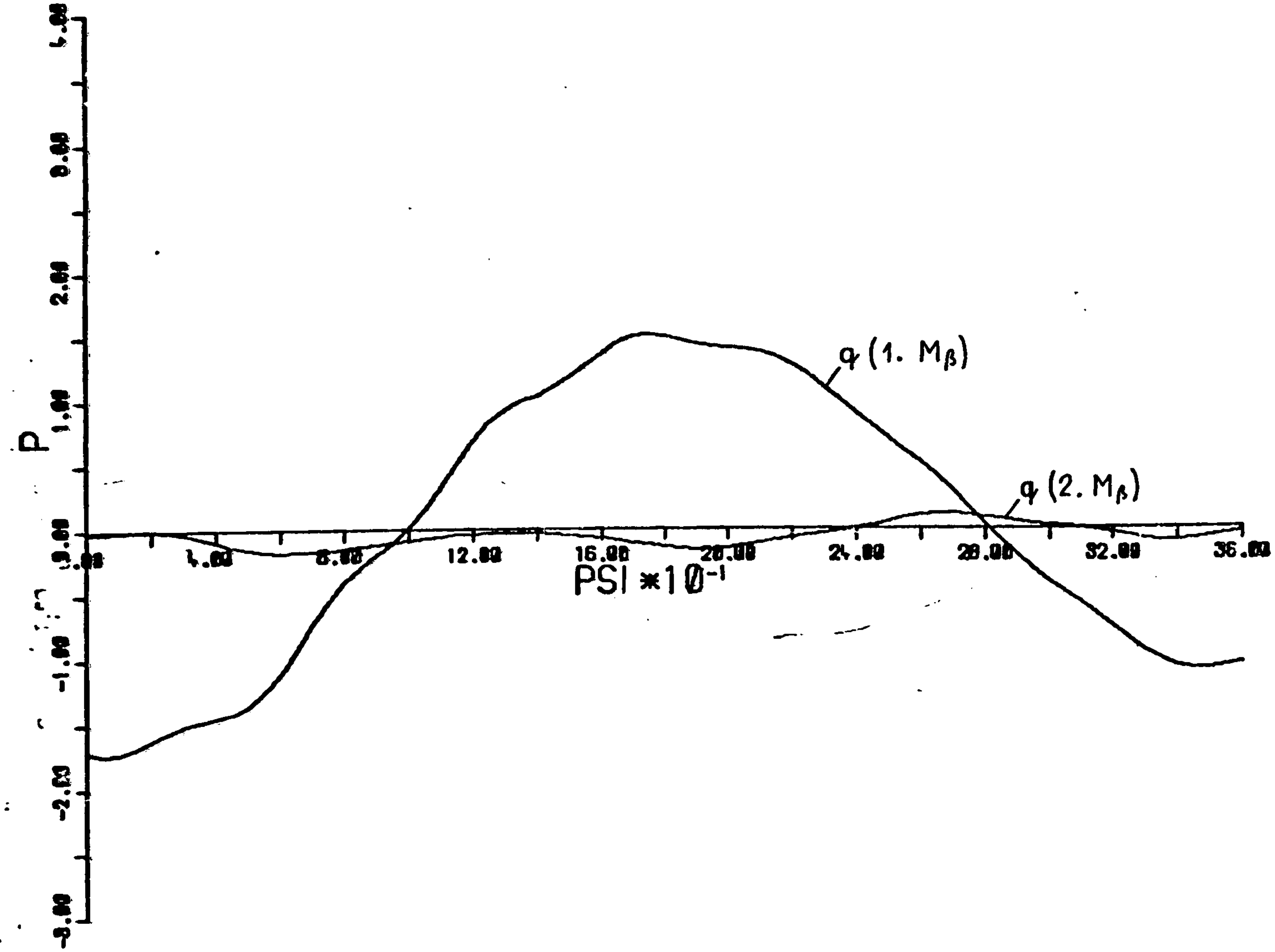

Fig. 8.18. Example IV, second rotation, $q(1.M_\beta)$ and $q(2.M_\beta)$.

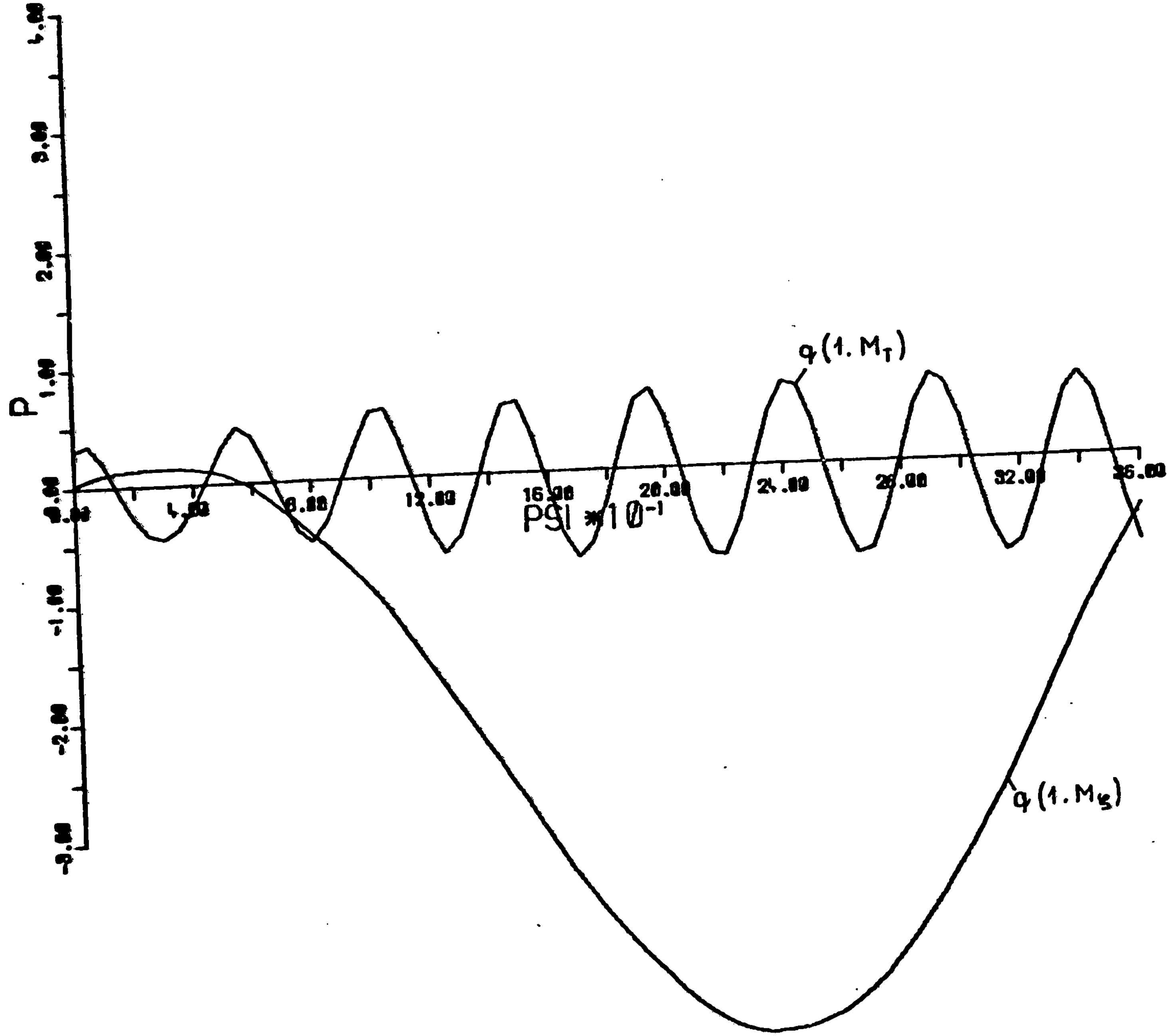

Fig. 8.19. Example IV, second rotation, $q(1.M_\zeta)$ and $q(1.M_T)$.

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